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ON THE STRONG CONVERSE OF THE CODING THEOREM FOR SYMMETRIC CHANNELS WITHOUT MEMORY[1]

BY

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1. Introduction. First we describe the channels we will discuss in this paper. The alphabet in which we send words contains just two letters, which we denote by 0, 1. Thus a sent word of length n is a sequence of n letters, each letter being a zero or a one. If we send the word $(x_1, \cdots, x_n)$, then the received word is $(Y_1, \cdots, Y_n)$, where $Y_1, \cdots, Y_n$ are independent chance variables, the probability distribution of Y_i depending only on the value of the parameter x_i. If there are just two possible values for Y_i, which we can assume are 0, 1, the channel is called a "binary channel." If Y_i has a probability density function whether x_i is zero or one, the channel is called a "semi-continuous channel." For a binary channel, $f_{x_i}(y)$ denotes $P(Y_i = y$, when the ith letter sent is $x_i)$. For a semi-continuous channel, $f_{x_i}(y)$ denotes the probability density function of Y_i when the ith letter sent is x_i.

A binary channel is called "symmetric" if $f_0(0) = f_1(1)$ (and therefore $f_0(1) = f_1(0)$). A semi-continuous channel is called "symmetric" if the probability distribution of $2[f_0(Y) + f_1(Y)]^{-1}f_0(Y)$ when Y has the probability density function $f_0(y)$ is the same as the probability distribution of $2[f_0(Y) + f_1(Y)]^{-1}f_1(Y)$ when Y has the probability density function $f_1(y)$.

A code of length L, word length n, and probability of error not greater than λ is a sequence of L pairs $(u_1, A_1), \cdots, (u_L, A_L)$ with the following properties:

(a) For each i, u_i is a sequence of zeros and ones, n symbols altogether;

(b) For each i, A_i is a collection of words of length n, each being a possible received word in the particular problem under consideration;

(c) For each i, when u_i is the sent word, $P((Y_1, \cdots, Y_n)$ is in $A_i) \geq 1 - \lambda$;

(d) The sets $A_1, \cdots, A_L$ are disjoint.

The use of such a code is well known. If the received word is in A_i, the receiver assumes that the word u_i was actually sent. Then, no matter which of the words $u_1, \cdots, u_L$ is sent, the probability that the receiver will be in error is not greater than λ.

A "converse to the coding theorem" is an inequality giving an upper bound for L, or, alternatively, for $\log_2 L$. (In this paper, each log is to the base e, unless another base is explicitly indicated.) Since several types of converse have appeared in the literature, it seems worthwhile to classify them carefully. The following classification seems useful.

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C denotes the capacity of the channel as defined in Feinstein [2]. A "weak converse" states the following. For any fixed $\epsilon > 0$, there is a positive value Z, such that there cannot exist a code of length $2^{n(C+\epsilon)}$ and probability of error not greater than Z, for large n. A "strong converse" states the following. For any fixed $\epsilon > 0$, and any fixed λ in the open interval $(0, 1)$, there cannot exist a code of length $2^{n(C+\epsilon)}$ and probability of error not greater than λ, for n large. A "stronger converse" states the following. For any fixed λ in the open interval $(0, 1)$, there is a constant K_λ such that there cannot exist a code of length $2^{nC+K_\lambda n^{1/2}}$ and probability of error not greater than λ, for large n.

Clearly, each of the converses listed is stronger than its predecessors. An interesting weak converse applicable to many types of channels is given in Theorem 5 of [3]. Wolfowitz [4] has proved the stronger converse for a binary channel, with K_λ a positive constant, and [5] has proved the strong converse for a semi-continuous channel. It is the purpose of the present paper to show that for both binary symmetric channels and semi-continuous symmetric channels, the stronger converse can be proved with K_λ a negative constant if $\lambda < \frac{1}{2}$. This makes the converse still stronger.

We define β_λ by the equation

$$\int_{-\infty}^{\beta_\lambda} (2\pi)^{-1/2} \exp\left(-\tfrac{1}{2}t^2\right) dt = 1 - \lambda.$$

Then if $\lambda < \frac{1}{2}$, $\beta_\lambda > 0$.

2. The binary symmetric channel. We denote the common value of $f_0(0)$ and $f_1(1)$ by q, and $1 - q$ by p. We assume that q is in the open interval $(\frac{1}{2}, 1)$. The capacity C for this channel is $1 + p \log_2 p + q \log_2 q$.

We have the following theorem. *For any $\delta > 0$, there is a number n_δ such that if $n > n_\delta$, then the length L of any code of word length n and probability of error not greater than λ must satisfy the inequality* $\log_2 L < nC - n^{1/2} [\beta_\lambda (pq)^{1/2} \log_2(q/p) - \delta]$.

Proof. If the word u_i is sent, the most probable received word is u_i itself, with probability q^n; the $\binom{n}{1}$ words differing from u_i in exactly one letter are tied for next most probable received word, each having probability pq^{n-1}; the $\binom{n}{2}$ words differing from u_i in exactly two letters are tied for next most probable received word, each having probability p^2q^{n-2}; etc.

We define K as the largest integer such that

$$\sum_{j=0}^{K} \binom{n}{j} p^j q^{n-j} \leq 1 - \lambda. \tag{1}$$

Denote by M_i the number of words in A_i. Then we must have

$$M_i \geq \sum_{j=0}^{K} \binom{n}{j},$$

since even the $\sum_{j=0}^{K} \binom{n}{j}$ which are the most probably received words have a total probability no greater than $1 - \lambda$.

Since there are 2^n different words of length n, and since $A_1, \cdots, A_L$ are disjoint, we have

$$2^n \geq \sum_{i=1}^{L} M_i \geq L \sum_{j=0}^{K} \binom{n}{j}, \quad \text{or} \quad L \leq 2^n \Big/ \sum_{j=0}^{K} \binom{n}{j} < 2^n \Big/ \binom{n}{K}.$$

Since on the left-hand side of (1) we are summing binomial probabilities, and since the binomial distribution approaches the normal distribution as n increases, we have by the central-limit theorem that $K = np + \beta(n, \lambda)\,(npq)^{1/2}$, where for any given positive value Δ, a number n_Δ can be found such that if $n > n_\Delta$, then $\beta_\lambda - \Delta \leq \beta(n, \lambda) \leq \beta_\lambda + \Delta$. We denote $\beta(n, \lambda)\,(pq)^{1/2}$ by B. Then $K = np + Bn^{1/2}$, where for $n > n_\Delta$, $(\beta_\lambda - \Delta)\,(pq)^{1/2} \leq B \leq (\beta_\lambda + \Delta)\,(pq)^{1/2}$.

Now

$$1 \Big/ \binom{n}{K} = K!(n-K)!/n!\,,$$

and using Stirling's inequalities $r^r(2\pi r)^{1/2} \exp(-r) < r! < r^r(2\pi r)^{1/2} \exp(-r + 1/12r)$, we find

$$1 \Big/ \binom{n}{K} < \frac{K^K(2\pi K)^{1/2} \exp(-K+1/12K)(n-K)^{n-K}[2\pi(n-K)]^{1/2} \exp\{-n+K+1/12(n-K)\}}{n^n(2\pi n)^{1/2} \exp(-n)}. \tag{2}$$

Setting $K = np + Bn^{1/2}$, simplifying the right-hand side of (2), and taking logs, we get

$$\begin{aligned} -\log \binom{n}{K} &< \frac{1}{2} \log(2\pi n) + \frac{1}{12[npq + Bn^{1/2}(q-p) - B^2]} \\ &\quad + (np + Bn^{1/2} + \tfrac{1}{2}) \log(p + Bn^{-1/2}) + (nq - Bn^{1/2} + \tfrac{1}{2}) \log(q - Bn^{-1/2}). \end{aligned} \tag{3}$$

Expanding $\log(p + Bn^{-1/2})$ and $\log(q - Bn^{-1/2})$ around p and q respectively, and substituting into (3), we get

$$-\log \binom{n}{K} < n(p \log p + q \log q) - n^{1/2} B \log(q/p) + \Omega \log n, \tag{4}$$

where Ω is a positive constant which does not depend on n. Converting the logarithms in (4) to the base 2, we find

$$-\log_2 \binom{n}{K} < n(p \log_2 p + q \log_2 q) - n^{1/2}[B \log_2(q/p) - \Omega n^{-1/2} \log_2 n].$$

Then

$$\log_2 L < n(1 + p \log_2 p + q \log_2 q) - n^{1/2}[B \log_2(q/p) - \Omega n^{-1/2} \log_2 n].$$

If

$$n > n_\Delta\,, \quad \log_2 L < nC - n^{1/2}[\beta_\lambda(pq)^{1/2} \log_2(q/p) - \Delta(pq)^{1/2} \log_2(q/p) - \Omega n^{-1/2} \log_2 n].$$

This proves the theorem, since Δ can be taken as close to zero as desired, and $n^{-1/2} \log_2 n$ approaches zero as n increases.

3. A theorem of Cramér. Before discussing the semi-continuous channel, we quote a theorem of Cramér, [1], which will be used later.

$Z_1, Z_2, \cdots$ are independent, identically distributed chance variables, each with

probability density function $v(z)$, expectation zero, and finite positive variance σ^2. There is a value $A > 0$ such that

$$R(h) = \int_{-\infty}^{\infty} \exp(hz)v(z)\, dz \quad \text{converges for} \quad |h| < A.$$

$m(h)$ denotes the first derivative of $\log R(h)$ with respect to h, and $\sigma_1^2(h)$ denotes the second derivative of $\log R(h)$ with respect to h. We define A_1 as $\sup\{h \mid R(h) \text{ converges}\}$. From our assumption, A_1 is positive and may be infinite. M is defined as $\sigma^{-1} \lim_{h \to A_1 - 0} m(h)$. M is positive and may be infinite.

Theorem. For any value g in the open interval $(0, M)$, the equation $m(h) = \sigma g$ has a unique root $h(g)$, which is positive. $P(Z_1 + \cdots + Z_n \geq \sigma gn) =$

$$n^{-1/2}[\{h(g)\sigma_1(h(g))(2\pi)^{1/2}\}^{-1} + Q(n, g)] \exp\{-n[h(g)m(h(g)) - \log R(h(g))]\},$$

where for any values γ_1, γ_2 with $0 < \gamma_1 < \gamma_2 < M$, there is a positive finite value $B(\gamma_1, \gamma_2)$ with $n \mid Q(n, g) \mid < B(\gamma_1, \gamma_2)$ for all g in the closed interval $[\gamma_1, \gamma_2]$.

4. The semi-continuous symmetric channel. We assume that

$$\int_{-\infty}^{\infty} \tfrac{1}{2}(f_0(y) + f_1(y))[\log\{2[f_0(y) + f_1(y)]^{-1}f_0(y)\}]^2\, dy$$

exists. Also, for any value y for which $f_0(y) = f_1(y) = 0$, we consider $2[f_0(y) + f_1(y)]^{-1}f_i(y)$ as zero for $i = 0, 1$, and $0 \log^k 0$ is always to be considered as equal to zero, for any positive k. The capacity C for the semi-continuous symmetric channel is

$$\int_{-\infty}^{\infty} f_0(y) \log_2 \{2[f_0(y) + f_1(y)]^{-1}f_0(y)\}\, dy.$$

For any given sequence $(x_1, \cdots, x_n)$ of zeros and ones, $W(x_1, \cdots, x_n)$ denotes a subset of $(Y_1, \cdots, Y_n)$ space such that

(a) $P((Y_1, \cdots, Y_n) \text{ is in } W(x_1, \cdots, x_n)) \geq 1 - \lambda$, when the joint probability density function of $Y_1, \cdots, Y_n$ is $f_{x_1}(y_1) f_{x_2}(y_2) \cdots f_{x_n}(y_n)$;

(b) $P((Y_1, \cdots, Y_n) \text{ is in } W(x_1, \cdots, x_n))$ is minimized when the joint probability density function of $Y_1, \cdots, Y_n$ is

$$\prod_{i=1}^{n} [\tfrac{1}{2}(f_0(y_i) + f_1(y_i))],$$

subject to (a). Then it is clear that the probability in (a) will be exactly $1 - \lambda$. We want to find the value of the probability in (b), which we denote by P^*.

By the familiar Neyman-Pearson lemma, the region $W(x_1, \cdots, x_n)$ is given by the set of points $(Y_1, \cdots, Y_n)$ with

$$\prod_{i=1}^{n} \{2[f_0(Y_i) + f_1(Y_i)]^{-1}f_{x_i}(Y_i)\} \geq k,$$

where k is a properly chosen constant. Alternatively, $W(x_1, \cdots, x_n)$ is the set of points $(Y_1, \cdots, Y_n)$ with

$$\sum_{i=1}^{n} \log\{2[f_0(Y_i) + f_1(Y_i)]^{-1}f_{x_i}(Y_i)\} \geq \log k.$$

By the definition of the symmetry of the channel, the distribution of

$$\sum_{i=1}^{n} \log \{2[f_0(Y_i) + f_1(Y_i)]^{-1} f_{x_i}(Y_i)\}$$

when the joint probability density function of $Y_1, \cdots, Y_n$ is $\prod_{i=1}^{n} f_{x_i}(y_i)$ does not depend on the sequence $(x_1, \cdots, x_n)$. Therefore the value of $\log k$ does not depend on the sequence $(x_1, \cdots, x_n)$.

Also, the probability P^* does not depend on the sequence $(x_1, \cdots, x_n)$. To show this, it suffices to show that the distribution of $2[f_0(Y) + f_1(Y)]^{-1} f_0(Y)$ when Y has the probability density function $\frac{1}{2}(f_0(y) + f_1(y))$ is the same as the distribution of $2[f_0(Y) + f_1(Y)]^{-1} f_1(Y)$ when Y has the probability density function $\frac{1}{2}(f_0(y) + f_1(y))$. The tth moments of these expressions are respectively

$$\int_{-\infty}^{\infty} 2^{t-1}[f_0(y) + f_1(y)]^{-t+1} f_0^t(y)\, dy, \qquad \int_{-\infty}^{\infty} 2^{t-1}[f_0(y) + f_1(y)]^{-t+1} f_1^t(y)\, dy.$$

But these integrals are respectively the $(t - 1)$st moment of the chance variable $2[f_0(Y) + f_1(Y)]^{-1} f_0(Y)$ when Y has the probability density function $f_0(y)$, and the $(t - 1)$st moment of the chance variable $2[f_0(Y) + f_1(Y)]^{-1} f_1(Y)$ when Y has the probability density function $f_1(y)$, and therefore are equal. Since the moments determine the distribution of a bounded chance variable, the demonstration is completed.

Therefore neither $\log k$ nor P^* depends on the sequence $(x_1, \cdots, x_n)$, and it is no loss of generality to assume that $x_1 = \cdots = x_n = 0$. We denote

$$\int_{-\infty}^{\infty} f_0(y) \log \{2[f_0(y) + f_1(y)]^{-1} f_0(y)\}\, dy \quad \text{by} \quad H,$$

and

$$\int_{-\infty}^{\infty} f_0(y)(\log \{2[f_0(y) + f_1(y)]^{-1} f_0(y)\})^2\, dy - H^2 \quad \text{by} \quad J^2,$$

J being taken as positive. Then, by the central-limit theorem, $\log k$ is equal to $nH - \beta(n, \lambda)n^{1/2} J$, where $\beta(n, \lambda)$ has the properties described in Sec. 2.

We denote

$$\int_{-\infty}^{\infty} \tfrac{1}{2}[f_0(y) + f_1(y)] \log \{2[f_0(y) + f_1(y)]^{-1} f_0(y)\}\, dy \quad \text{by} \quad S,$$

and

$$\int_{-\infty}^{\infty} \tfrac{1}{2}[f_0(y) + f_1(y)](\log \{2[f_0(y) + f_1(y)]^{-1} f_0(y)\})^2\, dy - S^2 \quad \text{by} \quad \sigma^2, \sigma$$

being taken as positive. To find P^*, we use Cramér's theorem of Sec. 3, with $Z_i = \log \{2[f_0(Y_i) + f_1(Y_i)]^{-1} f_0(Y_i)\} - S$, and $\sigma g n = nH - nS - \beta(n, \lambda)n^{1/2} J$, so that $g = (H - S)/\sigma - \beta(n, \lambda)\, J/\sigma n^{1/2}$. First we must verify that g is in the open interval $(0, M)$. It is easily shown that $m(h)$ is equal to

$$-S + \frac{\int_{-\infty}^{\infty} [\frac{1}{2}(f_0(y) + f_1(y))]^{-h+1} f_0^h(y) \log \{2[f_0(y) + f_1(y)]^{-1} f_0(y)\}\, dy}{\int_{-\infty}^{\infty} [\frac{1}{2}(f_0(y) + f_1(y))]^{-h+1} f_0^h(y)\, dy},$$

from which it follows that $m(1) = H - S$, and that $m(h)$ is a strictly increasing function of h for positive h, except in the trivial case (which we exclude) where $f_0(y) = f_1(y)$ almost everywhere. It is easily verified that $H - S$ is positive. This proves that g is in the open interval $(0, M)$, at least for large values of n.

Our next task is to find $h(g)$. We note that $dm(h)/dh$ is continuous in a neighborhood of $h = 1$, and is equal to J^2 at $h = 1$. Therefore we have $m(h) = m(1) + (h - 1)J^2 + \epsilon(h - 1)$, where $(1/r)\ \epsilon(r)$ approaches zero as r approaches zero. The equation $m(h(g)) = \sigma g$ becomes

$$H - S + (h(g) - 1)J^2 + \epsilon(h(g) - 1) = H - S - \beta(n, \lambda)Jn^{-1/2},$$

or

$$h(g) = 1 - \beta(n, \lambda)/(n^{1/2}J) + \delta_n ,$$

where $n^{1/2}\ \delta_n$ approaches zero as n increases.

Substituting this value of $h(g)$ in Cramér's theorem, we find that $\log P^* = -n [H - \beta(n, \lambda)\ Jn^{-1/2} + \Omega_n]$, where $n^{1/2}\ \Omega_n$ approaches zero as n increases. Denote by J' the quantity that J becomes when logarithms to the base 2 are used in the definition instead of logarithms to the base e. Then $\log_2 P^* = -n[C - \beta(n, \lambda)\ J'n^{-1/2} + D_n]$, where $n^{1/2}D_n$ approaches zero as n increases.

If we have a code $(u_1, A_1), \cdots, (u_L, A_L)$ of length L and probability of error not greater than λ, it follows from our discussion that $P((Y_1, \cdots, Y_n) \text{ is in } A_i) \geq P^*$, when the joint probability density function of $Y_1, \cdots, Y_n$ is $\prod_{i=1}^{n} [\frac{1}{2}(f_0(y_i) + f_1(y_i))]$. Since $A_1, \cdots, A_L$ are disjoint, it follows that $LP^* \leq 1$, or that $\log_2 L \leq n[C - \beta(n, \lambda)\ J'n^{-1/2} + D_n]$, where $n^{1/2}\ D_n$ approaches zero as n increases. Thus the stronger converse is proved for the semi-continuous symmetric channel.

5. Acknowledgment. The author would like to thank Professor J. Wolfowitz for many helpful discussions of coding theory.

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THE RAYLEIGH-RITZ METHOD FOR DISSIPATIVE OR GYROSCOPIC SYSTEMS*

BY

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1. Introduction. According to a well known principle of Raleigh, the natural frequencies of a conservative mechanical system can not be decreased when constraints are imposed on the system [1]. This principle is the basis for the Rayleigh-Ritz method for the approximate determination of the natural frequencies.

Rayleigh treated vibration about *static equilibrium*, but he did not consider vibration about *steady motion*. Thus, his analysis does not apply to vibration of machines with electric motors. The difficulty is that gyroscopic forces have a different nature from forces arising from a potential.

In a previous paper, the writer treated highly dissipative systems [2]. A relation was introduced termed "overdamping", which insures that the general solution of the equations of motion is a sum of exponential decays. The decay values of overdamped systems are analogous to the natural frequencies of conservative systems. It was shown that Rayleigh's principle applies to the decay values.

In this paper, a simple transformation is introduced, which converts the equations of motion for a conservative system into the equations of motion for an overdamped system. Moreover, the addition of gyroscopic forces does not change this correspondence. By means of this device, it is seen that Rayleigh's principle is valid for the natural frequencies of vibration of such conservative gyroscopic systems.

2. Vibrations of conservative systems about static equilibrium. An example of the type of system considered by Rayleigh, is the triple pendulum, shown in Fig. 1.

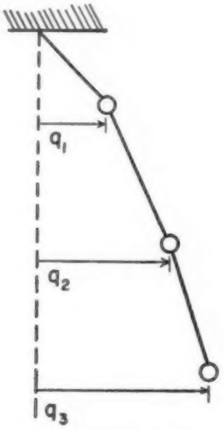

FIG. 1. Triple pendulum.

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The potential energy is of the form $c(q)/2$, where

$$c(q) = C_{11}q_1^2 + C_{22}q_2^2 + C_{33}q_3^2 + 2(C_{12}q_1q_2 + C_{23}q_2q_3 + C_{31}q_3q_1).$$

Of course, terms of higher order are neglected. The force components are obtained by differentiating the potential energy, so Newton's equations for small free vibrations are

$$A_iq_i'' + \sum_1^3 C_{ij}q_j = 0.$$

Here, A_1, A_2, and A_3, are the masses of the bobs, and a prime denotes differentiation with respect to the time, t.

In matrix notation, the equations of motion may be written as

$$Aq'' + Cq = 0. \tag{1}$$

In the general case of n degrees of freedom, q denotes a vector of n components, and A and C are real symmetric matrices with positive definite quadratic forms $a(q)$, and $c(q)$. Otherwise, A and C are arbitrary.

A normal mode of vibration is a solution of (1) of the form

$$q = ue^{i\omega t}. \tag{2}$$

Here, u is a constant vector, and the constant ω is termed a *natural frequency*. The equation which determines the normal modes of vibration is

$$-\omega^2Au + Cu = 0. \tag{3}$$

A *constraint* is a linear homogeneous relation imposed on the coordinates such as

$$q_1 - 2q_2 + q_3 = 0. \tag{4}$$

For the triple pendulum, this constraint could be achieved physically by replacing the two strings connecting the bobs by a rigid rod.

To study the effect of constraints on the spectrum of natural frequencies, Rayleigh [1], introduced the functional

$$p^2(q) = \frac{c(q)}{a(q)}. \tag{5}$$

The stationary values of p are precisely the natural frequencies ω_i. The stationary values of p when q is restricted by (4) give the natural frequencies ω_i' of the constrained system. An additional constraint imposed leads to the frequencies ω_i'' of a doubly constrained system. The qualitative relationships are brought out in Fig. 2. Considering only the positive frequencies, we see that the constrained spectrum separates the original spectrum (This was proved independently by Routh, [3]).

3. Overdamped systems. The presence of dissipative forces can be accounted for by adding an appropriate term to (1). Then, the equation of motion becomes

$$Aq'' + Bq' + Cq = 0. \tag{6}$$

Here, B is a symmetric matrix whose quadratic form $b(q')$ gives the rate of dissipation of energy [1].

Attention is now focused on systems where the dissipation is large. This would be the case, for example, if the triple pendulum were immersed in a very viscous fluid.

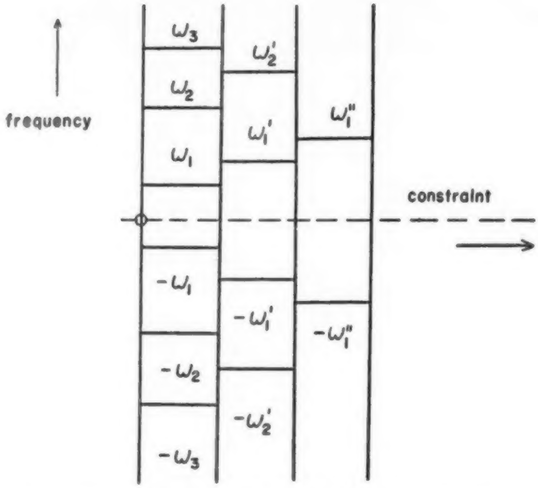

FIG. 2. Effect of constraints on the frequency spectrum.

The precise mathematical condition to be assumed is that the quadratic forms satisfy the following additional restriction

$$b^2(v) - 4a(v)c(v) > 0 \tag{7}$$

for an arbitrary real vector $v \neq 0$. In a previous paper, this was termed the *overdamping condition* [2]. It was shown that such onerdamped systens have *normal modes of decay.*

$$q = ue^{kt}. \tag{8}$$

Here, u is a constant vector, and the real constant k may be termed a *decay value.*

Substituting (8) in (6) gives

$$k^2Au + kBu + Cu = 0. \tag{9}$$

Forming the scalar product of (9) with u gives

$$\begin{gathered} k^2a + kb + c = 0, \text{ and} \\ k = [-b \pm (b^2 - 4ac)^{1/2}]/(2a) \end{gathered} \tag{10}$$

It was found that n of the normal modes require the positive choice of the radical in (10). These modes are termed *primary,* and their decay values are $k_1, k_2, \cdots, k_n$. Likewise, n normal modes require the negative sign. These modes are termed *secondary,* and their decay values are $h_1, h_2, \cdots, h_n$.

The following variational principle may be seen to apply to overdamped systems. Let the *primary functional* be defined as

$$p(v) = [-b(v) + \{b^2(v) - 4a(v)c(v)\}^{1/2}]/[2a(v)]. \tag{11}$$

Then the primary decay values are the stationary values of $p(v)$. Let the *secondary functional* be defined as

$$s(v) = [-b(v) - \{b^2(v) - 4a(v)c(v)\}^{1/2}]/[2a(v)]. \tag{12}$$

Likewise, the secondary decay values are the stationary values of $s(v)$.

A constraint on an overdamped system is seen to maintain condition (7). Thus the constrained system is also overdamped. The influence of constraint on the decay spectrum is illustrated in Fig. 3. A constraint is indicated by a prime. This diagram brings out the fact that the primary decay values of the constrained system separate the primary decay values of the original system. A similar separation relation holds for the secondary spectra.

It is seen from Fig. 3 that the following inequalities hold

$$h_1 \le s(v) \le h_3 < k_1 \le p(v) \le k_3 < 0. \tag{13}$$

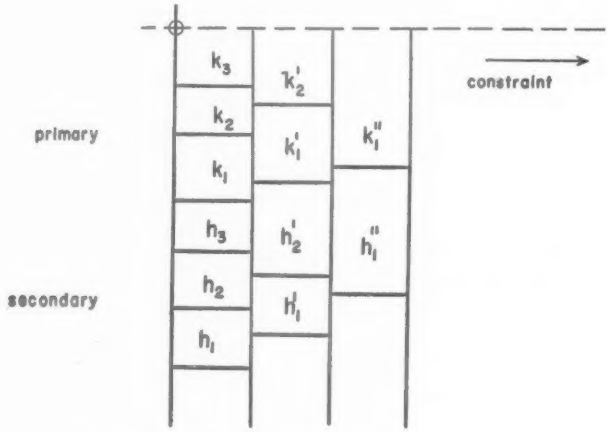

FIG. 3. Effect of constraints on the decay spectrum.

Here, v is an arbitrary vector; it results from a double constraint. Clearly, (13) would be useful for estimating the decay values of the overdamped triple pendulum.

These considerations show that the analogue of Rayleigh's principle holds for an overdamped system, provided the primary spectrum and the secondary spectrum are considered separately. Of course, even in the classical case, shown in Fig. 2, it is necessary to distinguish between the positive frequency spectrum and the negative frequency spectrum. This analogy is not accidental. Thus, by a trivial relaxation in the definitions, it is now to be shown that a conservative system is a special case of an overdamped system.

A *relative overdamped system* is defined by matrices A, B, and C, such that:

(i) They are symmetric.
(ii) The overdamping condition (7) holds.
(iii) A is positive definite.

With A, B, and C having these properties, let $k = k_0 + m$, in Eq. (9). Then (9) becomes $k_0^2 Au + k_0 B_0 u + C_0 u = 0$. Here $B_0 = 2mA + B$, and $C_0 = m^2 A + mB + C$. It may be verified that

$$b_0^2 - 4ac_0 = b^2 - 4ac > 0.$$

Moreover, if m is a sufficiently large positive constant, both B_0 and C_0 are positive definite. It follows that A, B_0, and C_0 define an "absolute" overdamped system. Thus, the spectrum of a relative overdamped system is simply the spectrum of an absolute overdamped system shifted upward.

The normal modes of vibration of a conservative system are defined by Eq. (3). But, clearly, the matrices A, 0, $-C$, correspond to a relative overdamped system. This proves that the conservative system is a special case.

4. Vibrations of conservative gyroscopic systems. Now of concern are mechanical systems whose small motions are governed by an equation of the form

$$Aq'' + Bq' + Cq = 0.$$

The matrices are to satisfy the conditions:

(i) A and C are symmetric positive definite.
(ii) B is antisymmetric.

We term such systems *regular*. It is seen that *regular* systems are conservative because the quadratic form $b(q')$ vanishes identically.

A simple example of a regular system is the gyroscopic pendulum shown in Fig. 4. A bob of mass m is spinning so as to have angular momentum L, directed along the rigid supporting rod of length h. The position of equilibrium is taken as origin of x, y coordinates in a horizontal plane.

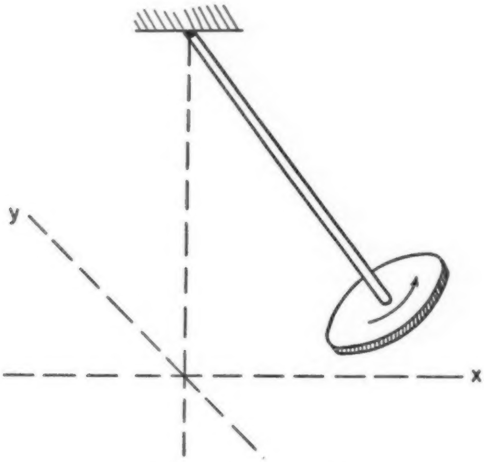

FIG. 4. Gyroscopic pendulum.

Taking moments about the point of support, leads to the following equations of motion.

$$mhx'' + Lh^{-1}y' + mgx = 0.$$
$$mhy'' - Lh^{-1}x' + mgy = 0.$$

It is easy to vertify that the normal modes of vibration are rotations in a circle with frequencies

$$\omega = \frac{L}{2mh^2} \pm \left[\left(\frac{L}{2mh^2} \right)^2 + \frac{g}{h} \right]^{1/2}.$$

The positive sign corresponds to counterclockwise rotation. A complete analysis of such pendulums is given by A. G. Webster [4]. General gyroscopic systems are treated by E. T. Whittaker [5].

Returning to the general case, it is seen that the equation for the normal modes of vibration of a regular system is

$$-\omega^2 Az + i\omega Bz + Cz = 0. \tag{14}$$

Here, the vector z may be complex, but it can be written as $z = u + iv$, where u and v are real vectors. This expression for z is substituted in (14), and the real and imaginary terms are separated. The two equations resulting, may be written as the following single equation, using compound matrices.

$$\left[\omega^2 \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \omega \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} - \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \right] \begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{15}$$

The matrices in (15) are symmetric. The overdamping condition is satisfied because

$$4(u \cdot Bv)^2 + 4[a(u) + a(v)][c(u) + c(v)] > 0.$$

Moreover, the first matrix in (15) is positive definite. Thus, (15) may be regarded as the equation for the normal modes of an overdamped system with $2n$ degrees of freedom.

If ω and (u, v) satisfy (15), then ω and $(v, -u)$ also satisfy (15). Thus, each decay value of (15) is "double". Otherwise, the decay spectrum of (15) is identical with the frequency spectrum of (14).

A constraint on (14), corresponds to two constraints on (15), one on u, and one on v. Or course, the constrained spectrum of (15) is double valued. Then it is seen that inequalities of the type (13) lead to the following separation property. The double values of the doubly constrained spectrum separate the double values of the original spectrum.

By the correspondence between the spectra of (14), and (15), it follows that the separation property holds for the natural frequencies of (14). Moreover, if ω is a natural frequency, it is seen from (14), that $-\omega$ is also. Thus, the frequency spectrum is exactly as shown in Fig. 2. This proves Rayleigh's principle for regular systems.

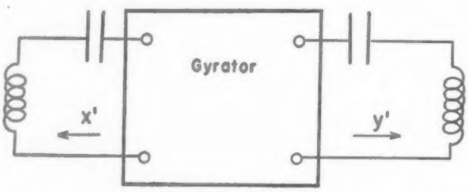

FIG. 5. Network analogue of the gyroscopic pendulum.

5. Non-reciprocal electrical networks. The classical analogy between mechanical systems and electrical networks makes force correspond to voltage, and velocity cor-

respond to current. The matrices A, B, and C, correspond to matrices of inductance, resistance, and $(\text{capacitance})^{-1}$. If A, B, and C, are symmetric, the network obeys the reciprocal law of Rayleigh [1].

A recent development in the network art has been networks which are the analogue of conservative gyroscopic systems. To aid in the analysis and synthesis of such non-reciprocal networks, Tellegen [6], introduced an ideal network element termed a gyrator. This concept is brought out in Fig. 5, which gives a network analogue of the gyroscopic pendulum.

It is seen that the gyrator serves as a representation of the antisymmetric B matrix.

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BOOK REVIEWS

Mathematical programming and electrical networks. By Jack B. Dennis. Technology Press of The Mass. Institute of Technology, John Wiley & Sons, Inc., New York, Chapman & Hall, Ltd., London, 1959. vi + 186 pp. $4.50.

This well-written and well-organized little volume, the third in the MIT Research Monograph Series, presents the results of the novel and recent researches of the author in establishing an equivalence between mathematical programming, particular linear and quadratic, and electrical networks composed of voltage and current sources, ideal diodes, resistors, and transformers.

The central idea is based on the extremum principle that in a circuit of current and voltage sources, resistors, and transformers the currents and voltages in the various branches of the network are such as to minimize the heating power losses. This is essentially a minimization of a quadratic form in the currents when resistors are present and a linear form when no resistors are present. Additional linear constraints are provided by the Kirchoff node laws. The introduction of diodes into the circuit imposes a condition of non-negativity on the currents. Thus, with resistors one has the electrical equivalent of a quadratic programming problem and without them, a linear programming problem.

The first two chapters and the Appendix, which occupies the last third of the book, provide some general mathematical notions for mathematical programming, local and global minima, concavity and convexity, Lagrange multipliers, the Legendre transformation, duality, and a number of pertinent theorems. In view of the greater advantages, in speed, economy, accuracy, and facility, of the use of large scale digital computers in solving these problems, as the author has pointed out, it is unlikely that anyone would use these techniques to solve a practical problem. Nevertheless, this research provides essentially a new outlook and insight into mathematical programming which has heretofore relied almost completely on economic, scheduling, and allocation examples as the vehicles for the introduction to mathematical programming. The minimum employment of the economic language and examples so common to other introductions to linear programming should be to most engineers' liking.

Chapter Three describes the properties of the various electrical devices of the networks, the network laws, and the equivalences of the various network problems. This possibly should appear a little earlier so that the non-engineer can see what the author is driving at. Chapter Four presents a diode-source algorithm for solving electrical models of certain network flow problems. For the maximum flow problem this is similar to the Ford-Fulkerson algorithm for transportation problems. The central notion here, useful throughout the remainder of the book, is that of a breakpoint curve, a curve of possible voltage-current combinations for a diode. The algorithm, elaborated further in Chapters Five and Six, consists in following this curve through in a manner analogous to the consecutive steps of the simplex method of Dantzig. Certain modifications to the algorithm and several other electrical algorithms called "valve" and "by-pass" algorithms are also presented and applied to different problems.

The final chapter proposes an iterative method for solving general equality- and inequality-constrained minimization problems based on a quadratic program. This method does not appear to differ widely from some methods in current practice. Errors are very few and relatively minor in consequences. The bibliography is adequate.

M. L. Juncosa

An introduction to statistical mechanics. By J. S. R. Chisholm and A. H. de Borde. Pergamon Press, Inc., New York, London, Paris, Los Angeles, 1958. ix + 160 pp. $6.00.

This book is designed as a text suitable for an Honours course in British Universities. The subject matter is limited to equilibrium phenomena and the applications are those which can be found in most books on statistical mechanics written during the last twenty-five years or so. The main virtues of the book are that it is short and clearly written. The scope, however, is very limited and there is no indication of the modern trends of the subject. The author claims to have a simplified version of the Darwin-

(Continued on p. 234)

FURTHER PROPERTIES OF CERTAIN CLASSES OF TRANSFER FUNCTIONS*

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Abstract. Some previously defined classes of rational transfer functions are extended to transfer functions that need not be rational. Several additional properties of these types of functions are then developed. Finally, bounds on certain derivatives of the unit impulse responses corresponding to such functions are shown to exist.

1. Let $Z(s)$ denote the transfer function of a system and let $W(t)$ be the response of the system to a unit impulse applied at time $t = 0$. Then $Z(s)$ and $W(t)$ are related by the Laplace transform.

$$Z(s) = \int_0^\infty W(t) \exp(-st)\, dt. \tag{1}$$

It will be assumed henceforth that $W(t)$ and its derivatives are integrable in any finite interval. The transfer function $Z(s)$ is a function of the complex variable, $s = \sigma + j\omega$. Since $W(t)$ is a real function for physical systems, the complex singularities of $Z(s)$ occur in complex conjugate pairs.

In a recent series of papers [1, 2, 3] a number of properties were established for certain classes of rational transfer functions. One of the objects of this paper is to remove the restrictions that $Z(s)$ be rational. This will require a more general form for the definitions of the classes of transfer functions. Let $Z(s)$ be analytic for $\sigma \geq 0$ and let it be asymptotic to K/s^k as s approaches infinity where K is a positive constant and k is a positive integer. In the special case where $Z(s)$ is rational, we have

$$Z(s) = K \frac{s^n + a_{n-1}s^{n-1} + \cdots + a_0}{s^m + b_{m-1}s^{m-1} + \cdots + b_0} \tag{2}$$

$$k = m - n > 0. \tag{3}$$

As noted above, $W(t)$ assumes only real values. Consequently, all the coefficients in (2) must be real.

Returning to the general case where $Z(s)$ need not be rational, let $Z_q(s)$ denote the following successive integrations of $Z(s)$.

$$Z_q(s) = \int_s^\infty ds_{q-1} \int_{s_{q-1}}^\infty ds_{q-2} \cdots \int_{s_1}^\infty Z(s_0)\, ds_0\,. \tag{4}$$

Only the principal branch of this multivalued function will be needed. In fact, it will be sufficient to assume that the arbitrary paths of integration in (4) never enter into the region defined by $\sigma \leq \sigma_1 < 0$ where σ_1 is the largest real part among the singularities of Z(s). Under this restriction, (4) can be considered a single-valued function.

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Also, let the real and imaginary parts of some subsequently needed quantities be denoted by

$$s = \sigma + j\omega, \tag{5}$$

$$s_q = \sigma_q + j\omega_q , \tag{6}$$

$$Z(j\omega) = R(\omega) + jI(\omega), \tag{7}$$

$$Z_q(j\omega) = R_q(\omega) + jI_q(\omega). \tag{8}$$

The following relations may then be obtained from (4). For q odd,

$$R_q(\omega) = (-1)^{(q+1)/2} \int_{\omega}^{\infty} d\omega_{q-1} \int_{\omega_{q-1}}^{\infty} d\omega_{q-2} \cdots \int_{\omega_1}^{\infty} I(\omega_0)\, d\omega_0 \tag{9}$$

and, for q even,

$$R_q(\omega) = (-1)^{q/2} \int_{\omega}^{\infty} d\omega_{q-1} \int_{\omega_{q-1}}^{\infty} d\omega_{q-2} \cdots \int_{\omega_1}^{\infty} R(\omega_0)\, d\omega_0 . \tag{10}$$

$R_q(\omega)$ and $I_q(\omega)$ are even and odd functions of ω, respectively [2; lemma 2].

We may now state the following definition.

Definition 1. *A function $Z(s)$ will be called a class k function if the following conditions hold.*

(a) *$Z(s)$ is analytic for $\sigma \geq 0$.*
(b) *$Z(s) \sim K/s^k$ as $s \to \infty$ where K is a positive constant and k is a positive integer.*
(c) *$Z_{k-1}(s)$ is a positive real function in the half plane, $\sigma \geq 0$.*

A function is said to be positive real if its real part is nonnegative for $\sigma \geq 0$ and if it assumes only real values along the real axis [4].

A certain subclass of each class k will be required in the following discussion. The definition of this subclass is again a generalization of the previously given one [2] in that the condition that $Z(s)$ be rational is dropped. Furthermore, $R(\omega)$ and $dI/d\omega$ are now allowed to assume the value of zero at $\omega = 0$.

Definition 2. *A function $Z(s)$ will be called a subclass k function if conditions* (a) *and* (b) *in Definition* 1 *hold and if the following condition holds.*

(c′) *For k odd, $R(\omega)$ has $k - 1$ changes of sign for $-\infty < \omega < \infty$ and $R(\omega)$ is positive in the neighborhood of $\omega = 0$; for k even, $I(\omega)$ has $k - 1$ changes of sign for $-\infty < \omega < \infty$ and $dI/d\omega$ is negative in the neighborhood of $\omega = 0$.*

It has been shown formerly [2; theorem 2] that all subclass k functions are class k functions. Even with these more general definitions, the previously given proof applies in precisely the same way. Furthermore, all the theorems in Part II of Ref. [2] and in Ref. [3] continue to hold since their proofs did not make use of the rationality of $Z(s)$. (Theorem 6 of Ref. [3] must be modified slightly if it is to apply in this case.)

One particular result [2; theorem 9] that the unit impulse response corresponding to any class k function must satisfy is

$$| W(t) | \leq \frac{Kt^{k-1}}{(k-1)!}. \tag{11}$$

The principal objective of this paper is to show that the following bounds exist on a

number of the derivatives of those $W(t)$ corresponding to subclass k functions. In particular, for $k \geq 2$

$$| W^{(\mu)}(t) | \leq \frac{K t^{k-\mu-1}}{(k - \mu - 1)!} \tag{12}$$

where the integer μ is restricted by $0 \leq \mu \leq (k - 1)/2$ if k is odd and by $0 \leq \mu \leq k/2$ if k is even. In developing this result, some new properties of the subclass k functions will be obtained.

2. The following lemmas will be needed.

Lemma 1. *Let $F(s)$ satisfy conditions* (a) *and* (b) *of Definition* 1. *Then, for k odd, $R(\omega)$ must have at least $k - 1$ changes of sign in $- \infty < \omega < \infty$ and, for k even, $I(\omega)$ must have at least $k - 1$ changes of sign in $- \infty < \omega < \infty$.*

The proof of this lemma has been given previously [2; lemma 3]. It applies in the case where $F(s)$ is not rational since the rational property was not invoked anywhere in its proof.

Lemma 2. *Let k be even and let $Z(s)$ be a subclass k function. Then $sZ(s)$ is a subclass $(k - 1)$ function.*

Proof. By condition (c′), $I(\omega)$ has $k - 1$ changes of sign in $- \infty < \omega < \infty$. Since $I(\omega)$ is an odd function [2; lemma 2], one of these sign changes occurs at $\omega = 0$. Now the real part of $sZ(s)$ equals $- \omega I(\omega)$ and, consequently, it has $(k - 2)$ sign changes in $- \infty < \omega < \infty$. Since $dI/d\omega$ is negative in the neighborhood of $\omega = 0$, $- \omega I(\omega)$ is positive in the same neighborhood. Thus, $- \omega I(\omega)$ fulfills condition (c′). Conditions (a) and (b) are also fulfilled by $sZ(s)$ so that the lemma is established.

Theorem 1. *Let $k \geq 3$ and let $Z(s)$ be a subclass k function. Then,*

$$G(s) = \int_s^\infty s_0 Z(s_0)\, ds_0 \tag{13}$$

is a subclass $(k - 2)$ function.

Proof. The theorem will first be established in the case where k is odd. Denoting the real part of $G(j\omega)$ by $R_G(\omega)$, we have

$$R_G(\omega) = -\int_\omega^\infty \omega_0 R(\omega_0)\, d\omega_0 \; . \tag{14}$$

Since $sZ(s)$ is analytic for $\sigma \geq 0$ and since $Z(s) = 0\,(1/s^k)$ as $s \to \infty$, (14) assumes the value of zero when the lower limit is set equal to $- \infty$. Hence,

$$R_G(\omega) = \int_{-\infty}^\omega \omega_0 R(\omega_0)\, d\omega_0 \; . \tag{15}$$

Integrating (15) by parts, the value of $R_G(\omega)$ at $\omega = 0$ may be expressed as

$$R_G(0) = -\int_{-\infty}^0 d\omega_1 \int_{-\infty}^{\omega_1} R(\omega_0)\, d\omega_0 \; .$$

Because $Z(s)$ is a subclass k function, $\int_{-\infty}^{\omega} R(\omega_0)\, d\omega_0$ will have $k - 2$ changes of sign in $- \infty < \omega < \infty$ one of which is at the origin. Hence,

$$\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\omega_1} R(\omega_0)\, d\omega_0 \tag{16}$$

will have $k - 3$ sign changes in $-\infty < \omega < \infty$. It follows that, since $R(\omega)$ is positive in the neighborhood of $\omega = 0$, (16) is negative at $\omega = 0$ so that $R_G(0)$ is positive.

Now consider the case where $k = 4\nu - 1$ ($\nu = 1, 2, 3, \cdots$). The quantity $\omega R(\omega)$ and, consequently, $R_G(\omega)$ become ultimately positive as ω decreases indefinitely toward $-\infty$. Hence, for $R_G(0)$ to be positive, $R_G(\omega)$ can have only an even number of sign changes in $-\infty < \omega < 0$. However, $\omega R(\omega)$ has $(k - 1)/2$ sign changes in $-\infty < \omega < 0$ and this is an odd number. Since integrating according to (15) can never increase the number of sign changes in $-\infty < \omega < 0$, $R_G(\omega)$ must have less than $(k - 1)/2$ sign changes in $-\infty < \omega < 0$. Moreover, by lemma 1, $R_G(\omega)$ has at least $(k - 3)/2$ sign changes in $-\infty < \omega < 0$. Thus, $R_G(\omega)$ has exactly $(k - 3)/2$ sign changes in $-\infty < \omega < 0$.

This result coupled with the facts that $R_G(\omega)$ is even and $R_G(0)$ is positive shows that $G(s)$ satisfies the requirements of condition (c′). Conditions (a) and (b) are also fulfilled so that $G(s)$ is a subclass $(k - 2)$ function.

A similar argument may be applied in the case where $k = 4\nu + 1$ ($\nu = 1, 2, 3, \cdots$).

Now let k be even. As before, it can be shown that $I_G(\omega)$ is given by

$$I_G(\omega) = \int_{-\infty}^{\omega} \omega_0 I(\omega_0)\, d\omega_0 \,. \tag{17}$$

This quantity is continuous and an odd function of ω. Hence, $I_G(0) = 0$. Also, its derivative is negative in the vicinity of $\omega = 0$ since $I(\omega)$ has this property and $I(0) = 0$.

Since $Z(s)$ is a subclass k function, $\omega I(\omega)$ has $(k - 2)/2$ changes of sign within the interval $-\infty < \omega < 0$. Let ω_i be a point where $I_G(\omega)$ changes sign. Since $I_G(\omega)$ is related to $\omega I(\omega)$ according to (17), $I_G(\omega)$ must have a smaller number of changes of sign in $-\infty < \omega < \omega_i$ than $\omega I(\omega)$ does. Furthermore, the point $\omega = 0$ is a point where $I_G(\omega)$ changes sign so that $I_G(\omega)$ has no more than $(k - 4)/2$ changes of sign in $-\infty < \omega < 0$. Hence, invoking Lemma 1, $I_G(\omega)$ has exactly this number of sign changes in $-\infty < \omega < 0$. It follows that $G(s)$ is a subclass $(k - 2)$ function. This completes the proof.

Theorem 2. *Let $Z(s)$ be a subclass k function where $k \geq 2$. Also, let μ be any integer in the range $0 \leq \mu \leq (k - 1)/2$ if k is odd and let μ be any integer in the range $0 \leq \mu \leq k/2$ if k is even. Then $s^{\mu}Z(s)$ is a class $(k - \mu)$ function.*

Proof. If a function $F(s)$ is a class k function, then the quantity $(-1)^h d^h F/ds^h$ is a class $(k + h)$ function. This result is an immediate consequence of the definition of a class k function.

Let k be odd ($k = 2\nu + 1$; $\nu = 1, 2, 3, \cdots$). To establish the conclusion, it will be shown that the function,

$$\int_{s}^{\infty} ds_{\mu-1} \int_{s_{\mu-1}}^{\infty} ds_{\mu-2} \cdots \int_{s_1}^{\infty} s_0^{\mu} Z(s_0)\, ds_0 \,, \tag{18}$$

is a class $(k - 2\mu)$ function where $0 \leq \mu \leq \nu = (k - 1)/2$.

Consider the innermost integral of (18). Integrating by parts repeatedly we may write

$$\int_s^\infty s_0^\mu Z(s_0)\, ds_0 = s^{\mu-1} \int_s^\infty s_0 Z(s_0)\, ds_0 + (\mu - 1) \int_s^\infty s_1^{\mu-2}\, ds_1 \int_{s_1}^\infty s_0 Z(s_0)\, ds_0$$

$$= s^{\mu-1} \int_s^\infty s_0 Z(s_0)\, ds_0$$

$$+ (\mu - 1)s^{\mu-3} \int_s^\infty s_1\, ds_1 \int_{s_1}^\infty s_0 Z(s_0)\, ds_0$$

$$+ (\mu - 1)(\mu - 3) \int_s^\infty s_2^{\mu-4}\, ds_2 \int_{s_2}^\infty s_1\, ds_1 \int_{s_1}^\infty s_0 Z(s_0)\, ds_0\,.$$

Continuing this process of integrating by parts the last term in this sum, the following result may be obtained. For μ odd,

$$\int_s^\infty s_0^\mu Z(s_0)\, ds_0 = s^{\mu-1} \int_s^\infty s_0 Z(s_0)\, ds_0 + \cdots$$
$$+ (\mu - 1)(\mu - 3) \cdots 4\cdot 2 \int_s^\infty s_{(\mu-1)/2}\, ds_{(\mu-1)/2} \cdots \int_{s_1}^\infty s_0 Z(s_0)\, ds_0 \tag{19}$$

and, for μ even,

$$\int_s^\infty s_0^\mu Z(s_0)\, ds_0 = s^{\mu-1} \int_s^\infty s_0 Z(s_0)\, ds_0 + \cdots$$
$$+ (\mu - 1)(\mu - 3) \cdots 3\cdot 1\cdot \int_s^\infty ds_{\mu/2} \int_{s_{\mu/2}}^\infty s_{\mu/2-1}\, ds_{\mu/2-1} \cdots \int_{s_1}^\infty s_0 Z(s_0)\, ds_0\,. \tag{20}$$

Repeatedly integrating in the above manner, (18) may be rearranged into the following finite sum.

$$\int_s^\infty ds_{\mu-1} \int_{s_{\mu-1}}^\infty ds_{\mu-2} \cdots \int_{s_1}^\infty s_0^\mu Z(s_0)\, ds_0$$

$$= \int_s^\infty s_{\mu-1}\, ds_{\mu-1} \int_{s_{\mu-1}}^\infty s_{\mu-2}\, ds_{\mu-2} \cdots \int_{s_1}^\infty s_0 Z(s_0)\, ds_0$$

$$+ A_1 \int_s^\infty ds_\mu \int_{s_\mu}^\infty ds_{\mu-1} \int_{s_{\mu-1}}^\infty s_{\mu-2}\, ds_{\mu-2} \cdots \int_{s_1}^\infty s_0 Z(s_0)\, ds_0 \tag{21}$$

$$+ A_2 \int_s^\infty ds_{\mu+1} \int_{s_{\mu+1}}^\infty ds_\mu \int_{s_\mu}^\infty ds_{\mu-1} \int_{s_{\mu-1}}^\infty ds_{\mu-2} \int_{s_{\mu-2}}^\infty s_{\mu-3}\, ds_{\mu-3} \cdots \int_{s_1}^\infty s_0 Z(s_0)\, ds_0$$

$$+ \cdots\,.$$

In this expression, all the A_i are positive integers.

Now Theorem 1 may be applied repeatedly to each term on the right hand side of (21) to show that each term is a subclass $(k - 2\mu)$ function. That is, working from the innermost integral outward, Theorem 1 indicates that, when the integrand possesses the factor s, each integration yields a function of some subclass and the order of the resulting subclass is reduced by two from the order of the subclass of the initial function. Moreover, when the integrand does not have the factor s, each integration produces a subclass function whose order is reduced by one. Consequently, each term on the right hand side of (21) is a subclass $(k - 2\mu)$ function and, hence, a class $(k - 2\mu)$ function.

Furthermore, if $F(s)$ and $G(s)$ are class k functions and if A and B are positive

numbers, then it follows from the definition of the class k functions that $AF(s) + BG(s)$ is also a class k function. Therefore, (21) is a class $(k - 2\mu)$ function and, by the remarks in the first paragraph of this proof, $s^{\mu}Z(s)$ is a class $(k - \mu)$ function.

The same result may be established in the case when k is even. ($k = 2\nu$; $\nu = 2, 3, 4, \cdots$. Lemma 2 states the theorem in the case when $k = 2$.) The same procedure as used before may be applied to the expression,

$$\int_{s}^{\infty} ds_{\mu-2} \int_{s_{\mu-2}}^{\infty} ds_{\mu-3} \cdots \int_{s_1}^{\infty} s_0^{\mu} Z(s_0)\, ds_0\ , \tag{22}$$

to show that it is a class $(k - 2\mu + 1)$ function where $0 \le \mu \le \nu = k/2$. In this case (22) is rearranged into the following finite sum.

$$\begin{aligned} s \int_{s}^{\infty} s_{\mu-2}\, ds_{\mu-2} \int_{s_{\mu-2}}^{\infty} s_{\mu-3}\, ds_{\mu-3} \cdots \int_{s_1}^{\infty} s_0 Z(s_0)\, ds_0 \\ + B_1 \int_{s}^{\infty} ds_{\mu-1} \int_{s_{\mu-1}}^{\infty} s_{\mu-2}\, ds_{\mu-2} \cdots \int_{s_1}^{\infty} s_0 Z(s_0)\, ds_0 \\ + B_2 \int_{s}^{\infty} ds_{\mu} \int_{s_{\mu}}^{\infty} ds_{\mu-1} \int_{s_{\mu-1}}^{\infty} ds_{\mu-2} \int_{s_{\mu-2}}^{\infty} s_{\mu-3}\, ds_{\mu-3} \cdots \int_{s_1}^{\infty} s_0 Z(s_0)\, ds_0 \\ + \cdots . \end{aligned}$$

In this expression, the B_i are all positive integers. Moreover, each term is a subclass $(k - 2\mu + 1)$ function by the same argument as before. Hence, (22) is in class $(k - 2\mu + 1)$. Differentiating it $\mu - 1$ times and multiplying by $(-1)^{\mu-1}$ will produce a class $(k - \mu)$ function. This completes the proof.

3. The principal conclusion of this paper may now be stated.

Theorem 3. *Under the hypothesis of Theorem* 2,

$$| W^{(\mu)}(t) | \le K \frac{t^{k-\mu-1}}{(k - \mu - 1)!}.$$

Proof. The transfer function $Z(s)$ is asymptotic to K/s^k $(k \ge 2)$ as $s \to \infty$. Hence, by the initial value theorem [5; theorem 15, p. 267], the corresponding unit impulse response $W(t)$ and its first $k - 2$ derivatives all have an initial value of zero. Thus, for the stated ranges of μ, $s^{\mu}Z(s)$ is the Laplace transform of $W^{(\mu)}(t)$ [5; p. 129].

Finally, it has been shown [2; theorem 9], that the unit impulse response corresponding to a class k function is bounded according to (11). Consequently, the conclusion follows immediately from Theorem 2.

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ON THE NON-EXISTENCE OF TRANSONIC PERTURBATIONS*

BY

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M. Schäfer [2] has developed a practical method to construct plane transonic potential flows directly by introducing a mesh of derived characteristics so defined that the mesh is real in both the supersonic and subsonic regions and joins continuously on the sonic line. Since the general solution of the equations for derived characteristics leads to great mathematical complications, Schäfer considers a simple solution which leads to the well-known Chaplygin solutions of the gasdynamic equations. These are a one parameter family of solutions which by suitable choice of the parameter, can be made to describe flows past profiles of practical interest.

Since the equations are linear in the derived characteristic variables which, in the case of the simple solution mentioned above, coincide with the polar hodograph variables, it is possible to find new solutions by superimposing solutions with different values of the parameter. In a second paper Schäfer [3] discusses the perturbation of profiles corresponding to special values of the parameter by superimposing solutions with positive integral values of the parameter on these specially chosen solutions which may be regarded as basic flows. The superimposed solutions are multiplied by coefficients which can be arbitrarily chosen to satisfy perturbed boundary conditions. Schäfer studies perturbations of the boundary conditions which are given by analytic (exponential) functions but which are practically localized in the supersonic region.

C. S. Morawetz [1] on the other hand, shewed that no perturbations involving only a finite segment of the supersonic boundary of a plane potential flow exist. If the perturbations studied by Schäfer could be regarded as numerical approximations of strictly localized perturbations, we would indeed have a family of flows depending—not necessarily smoothly[1]—on a parameter and with the same velocity at infinity namely, zero, which differ only along a finite segment of the supersonic boundary. Moreover, the theorems of Morawetz cannot be applied directly to these flows since in general a direction singularity occurs at infinity and hence infinity cannot be represented by a point in the hodograph plane considered by Morawetz.

Firstly, we remark that if the superposition is truncated, the perturbations must have an analytic character since each term in the summation is an analytic function of its variables. This means that singularities of curvature are impossible and the perturbations cannot be restricted to a finite segment of the supersonic boundary. We shew that the coefficients in the infinite series cannot be so chosen that a singularity is introduced into the boundary.

Using the simple solution of the equations for derived characteristics mentioned above, Schäfer [2] obtained the one parameter family of solutions of the gasdynamic equations in terms of certain metric functions as dependent variables and the polar

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[1]A set of flows given by infinite series of the solutions with arbitrary coefficients may be constructed by summing over the parameter.

hodograph variables v and ϑ as independent variables, known as the Chaplygin solutions. In Schäfer's notation the solutions are [2]:

$$\Xi = \frac{k}{\rho} v^{k-1} H_k(v^2) \begin{cases} \sin k\vartheta & \text{or} \\ \cos k\vartheta \end{cases}, \tag{1a}$$

$$\mathrm{H} = \frac{1}{\rho} [k v^{k-1} H_k(v^2) + 2 v^{k+1} H_k'(v^2)] \begin{cases} \cos k\vartheta & \text{or} \\ \sin k\vartheta \end{cases}, \tag{1b}$$

$$x - x_0 = \frac{1}{\rho}\left[\frac{v^{k+1}}{k+1} H_k' \sin (k+1)\vartheta + \frac{v^{k-1}}{k-1} (kH_k + v^2 H_k') \sin (k-1)\vartheta\right], \tag{2a}$$

$$y - y_0 = \frac{1}{\rho}\left[\frac{-v^{k+1}}{k+1} H_k' \cos (k+1)\vartheta + \frac{v^{k-1}}{k-1} (kH_k + v^2 H_k') \cos (k-1)\vartheta\right], \tag{2b}$$

$$\psi = v^k H_k(v^2) \begin{cases} \cos k\vartheta & \text{or} \\ \sin k\vartheta \end{cases}, \tag{3}$$

where

$$H_k = F(\alpha, -\beta; \gamma; v^2), \text{ the hypergeometric function,}$$

and

$$H_k'(w) = \frac{dH_k(w)}{dw}.$$

Ξ and H are the metric functions giving the relation between the cartesian coordinates x and y in the field of flow, and the derived characteristics which in the case considered coincide with the polar hodograph variables v and ϑ (v is the absolute value of the velocity vector made dimensionless with the velocity at zero pressure). ψ is the stream function, ρ the density made dimensionless with the stagnation density and κ the adiabatic gas constant.

We shall require an asymptotic expression for the hypergeometric function, $H_k(v^2)$, for large positive values of k. We write

$$H_k = \sum_{i=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\gamma + i)} \frac{\Gamma(\alpha + i)}{\Gamma(\alpha)} \frac{\Gamma(i - \beta)}{\Gamma(-\beta)} \frac{v^{2i}}{i!}, \tag{4}$$

where

$$\alpha = \frac{1}{2}\left\{k - \frac{1}{\kappa - 1} + \left[k^2 \frac{\kappa + 1}{\kappa - 1} + \frac{1}{(\kappa - 1)^2}\right]^{1/2}\right\},$$

$$-\beta = \frac{1}{2}\left\{k - \frac{1}{\kappa - 1} - \left[k^2 \frac{\kappa + 1}{\kappa - 1} + \frac{1}{(\kappa - 1)^2}\right]^{1/2}\right\}.$$

and $\gamma = k + 1$. $\alpha > 0$ and $\beta > 0$ for positive k. We have for large k

$$\frac{\Gamma(\gamma)}{\Gamma(\gamma + i)} \sim \gamma^{-i}; \qquad \frac{\Gamma(\alpha + i)}{\Gamma(\alpha)} \sim \alpha^i$$

and

$$H_k(v^2) \sim \sum_{i=0}^{\infty} \frac{\Gamma(i - \beta)}{\Gamma(-\beta)} \frac{1}{i!}\left[\frac{1}{2} v^2\left(1 + \left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2}\right)\right]^i = \left[1 - \frac{1}{2} v^2\left(1 + \left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2}\right)\right]^{\beta}. \tag{5}$$

By suitable choices of k Schäfer makes the solutions (1) represent basic flows of practical interest. Perturbations are then applied to these basic flows by putting

$$\Xi(v, \vartheta) = \Xi^*(v, \vartheta) + \frac{1}{\rho} \sum_{k=k_1}^{\infty} v^{k-1} H_k(v^2)(a_k \cos k\vartheta - b_k \sin k\vartheta), \tag{6a}$$

$$\mathrm{H}(v, \vartheta) = \mathrm{H}^*(v, \vartheta) + \frac{1}{\rho} \sum_{k=k_1}^{\infty} (kv^{k-1}H_k + 2v^{k+1}H_k')(a_k \sin k\vartheta + b_k \cos k\vartheta). \tag{6b}$$

The symbols with stars denote quantities in the undisturbed flow and plain symbols quantities in the disturbed flow. Since the equations for Ξ and H are linear, the perturbed metric functions as defined by (6) are again solutions of these equations. It is clearly more convenient to satisfy the perturbed boundary conditions in the physical plane and the expressions for x and y can be supplemented accordingly since they are linear combinations of Ξ and H. We shall now shew that the perturbation terms in (6) must be analytic functions of v and ϑ. We state the proposition in the following form: suppose the perturbation terms in (6) converge everywhere in the field of flow except possibly in a finite number of isolated points in the supersonic region. Then these terms converge absolutely and uniformly in all domains and they have term by term derivatives of all orders with respect to both variables.

Consider the function

$$v^{k-1}H_k(v^2). \tag{7}$$

This function tends to zero with increasing k, for in (5) $v < 1$ and β is asymptotically of the form

$$bk + O(k^{-1/2}), \qquad b > 0.$$

The asymptotic expression for (7) is

$$\left\{v\left[1 - \frac{1}{2}v^2\left(1 + \left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2}\right)\right]^b\right\}^k = \{V(v)\}^k \tag{8}$$

with

$$b = \frac{1}{2}\left(\left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2} - 1\right).$$

V increases with v in the interval

$$0 \le v^2 < 2\left\{\left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2}\left[1 + \left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2}\right]\right\}^{-1}$$

In the latter point V has a maximum whereafter it decreases to the value zero for

$$v^2 = 2\left\{1 + \left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2}\right\}^{-1} = v_2^2\,.$$

We can therefore say that there is a limiting point of maxima of (7) at

$$v^2 = 2\left\{\left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2}\left[1 + \left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2}\right]\right\}^{-1} = v_1^2$$

with respect to positive k. Since the treatment of the perturbation term in (6) is the

same whether we regard it as a sine series, a cosine series or a mixed series, we shall simplify by putting $b_k = 0$ for all k. We now put

$$a_k = f(k)A^k$$

and derive restrictions on A and f such that the conditions of our proposition are satisfied. A is simply a normalizing factor for the expression (8). Let A be chosen such that

$$Av_A\left[1 - \frac{1}{2}v_A^2\left(1 + \left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right)\right]^b = 1, \qquad v_A \neq v_1 .$$

With v_A in the range $0 \leq v_A < v_2$, the perturbation term in (6) would diverge as cosine series for all values of v in the interval $v_A < v \leq v_1$ or $v_1 \leq v < v_A$ according as $v_A < v_1$ or $v_1 < v_A$[1]. If the conditions of our proposition are still to be satisfied we must therefore have $v_A = v_1$. On $f(k)$ we must then obviously impose the restriction

$$f(k) = o(A^k).$$

The perturbation term can have at most one singular point with respect to v *viz.*, the point $v = v_1$. The order of this singularity is determined by the form of $f(k)$. It follows that the perturbation term is a uniformly and absolutely convergent cosine series with v in any subinterval of the interval

$$0 \leq v^2 \leq 2\left\{1 + \left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right\}^{-1} = v_2$$

which excludes $v = v_1$. We exclude the range

$$2\left\{1 + \left(\frac{\kappa+1}{\kappa-1}\right)^{1/2}\right\}^{-1} < v^2 \leq 1$$

since in this range (8) becomes complex. We shall see that the mapping of the hodograph plane on the physical plane ceases to be unique at the point $v = v_2$.

We moreover know that each term of the cosine series is a uniformly and absolutely convergent power series of v in the range considered. Hence the perturbation term can be rearranged as a power series in v which has term by term derivatives of all orders with respect to v in all subintervals which do not include the point $v = v_1$.

Differentiating the perturbation term formally n times with respect to ϑ, we obtain

$$\Sigma(-)^t k^n f(k) A^k v^{k-1} H_k \begin{cases}\cos k\vartheta \\ \sin k\vartheta\end{cases} \quad \text{or, with} \quad t = [\tfrac{1}{2}n] + \tfrac{1}{2}.$$

But in this expression we still have

$$(-)^t k^n f(k) = o(A^k) \quad \text{since} \quad f(k) = o(A^k).$$

Hence the perturbation term also has term by term derivatives of all orders with respect to ϑ unless $v = v_1$. Note that if the perturbation term is singular at this value, then there will in general be a singular line in the supersonic region *viz.*, the isotach $v = v_1$.

We now shew that with the above restrictions on a_k our proposition also holds for (6b). The asymptotic expression for the coefficients of the perturbation term in (6b) is

[1]For we can find a k_0 for every such v such that $A^k v^{k-1} H_k \sim \eta^k$ for all $k > k_0$ with $\eta > 1$.

$$\left\{v\left[1 - \frac{1}{2}v^2\left(1 + \left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2}\right)\right]^b\right\}^k \left\{1 - \frac{1}{2}\left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2}\left[\left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2} + 1\right]v^2\right\}. \tag{9}$$

The first factor is the same as for (6b) while the second factor has a zero in the point $v = v_1$ where $V(v)$ has a maximum. (9) therefore increases steadily until it approaches $v = v_1$ where it drops off sharply to negative values with the same amplitude in the range $v_1 < v < v_2$. The coefficients of (6b) are thus seen to have a limiting point of maxima in $v = v_1$ with respect to k. This limiting point is clearly never reached and away from this point the coefficients of (6b) become asymptotically

$$\left\{Av\left[1 - \frac{1}{2}v^2\left[1 + \left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2}\right]\right]^b\right\}^k$$

which is the same as the equivalent expression for (6a). Hence there can be no singularity in the perturbation term in (6b).

If we require furthermore that singularities only be propagated along characteristics, we must also exclude the singularity in (6a) since an isotach cannot coincide with a characteristic.

In order to exclude the range $v_2 < v < 1$ we consider the mapping of the hodograph plane on the physical plane. The Jacobian of this mapping is (see [3])

$$\frac{\partial(x, y)}{\partial(v, \vartheta)} = \cdot\frac{1}{v}\left(\frac{c^2 - v^2}{c^2}\,\Xi^2 + \mathrm{H}^2\right). \tag{10}$$

By comparing (10) with (8) and (9) we see that the expression

$$\left[1 - \frac{1}{2}v^2\left(1 + \left(\frac{\kappa + 1}{\kappa - 1}\right)^{1/2}\right)\right]^b$$

is asymptotically a factor of the Jacobian so that the mapping will cease to be unique when $v = v_2$. Moreover a limiting line will generally have appeared in the basic flow before this velocity is reached—this is e.g., the case in the basic flow with $k = -\frac{5}{3}$ considered by Schäfer.

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BOOK REVIEWS

(Continued from p. 222)

Fowler method for obtaining the canonical distribution. The details of this, which are given in an appendix, are sound but the outline of it given in the text is deceptive. In essence the book is a condensed version of the books by Fowler.

G. H. Newell

Theory of value, an axiomatic analysis of economic equilibrium. By Gerard Debreu. John Wiley & Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1959. xii + 114 pp. $4.75.

This book is a landmark in the development of mathematical economics. Coordinating the work that has gone into the central body of economic theory since Cournot and recasting it in the mold of contemporary mathematics, this slender volume gives the definitive treatment of the Theory of Value constructed rigorously from first axioms.

Chapter 1 assembles all the mathematical concepts and propositions to be used in the economic chapters. Evidently a great deal of thought has gone into such organizational matters as notation and layout, and this pays off handsomely in clarity and readability throughout the book. The departure from traditional calculus-bound methods is apparent in this box of tools: set and point set theory, ordering, metric spaces, and topology including fixed point theory. Chapter 2 clarifies all economic concepts employed and will be particularly useful to the mathematical reader as a grammer of the economist's language. Economists on their part will be delighted at the crystal clarity with which the basic concepts and underlying assumptions are here exposed. The analysis gets under way in chapter 3 with a discussion of the theory of production. The axioms on technology are presented and discussed in order of decreasing plausibility. These include the technology assumed in Linear Programming models as a special case. The implications of profit maximization are studied and "comparative statics" is analyzed. The place of derivatives and of Lagrangian Multipliers is taken here by supporting hyperplanes to convex sets. The 11 pages of this analysis are the most compact complete treatment of the general theory of production in existence.

In Chapter 3 the theory of consumption is constructed analogously on the basis of complete, transitive, continuous preference orderings on closed, connected consumption sets. The existence of (ordinal) utility functions is proved and the theory of consumers' choice subject to wealth constraints is developed. The high point of the book is reached in Chapter 5 where the existence of an equilibrium for a "private ownership economy" characterized essentially by non-increasing returns in production and insatiable consumption subject to non-increasing marginal utility (these are old-fashioned economic terms not used here) is proved from Kakutani's fixed point theorem. Uniqueness and stability are deliberately excluded from this theory, presumable because the former is essentially trivial but messy and the latter leads into the vast area of dynamic processes where no orderly system is as yet in sight. Chapter 7 treats briefly the optimality of the equilibrium. This is followed in Chapter 8 by an extension of the model to a situation of uncertainty which interestingly enough does not involve any probabilistic considerations—and hence has no need of the apparatus of expected (cardinal) utility. This final chapter gives in effect a condensed summary of the entire theory of this book.

One can only marvel at the precision, the elegance, and the perfection of this work. Throughout, it is an original, penetrating statement of the heart of neoclassical economic theory, the theory of value and equilibrium, and it brings out as never before the unity and architectural beauty of this body of thought. Its impact will be felt by every mathematical economist. Without question this book will be a standard reference. Beyond this it should prove a boon to all mathematically educated persons in search of a royal road to economic theory.

With this latest volume the series of Cowles Foundation monographs—including such classics as "Activity Analysis of Allocation and Production" and "Statistical Inference in Dynamic Economic Models"—has achieved a new high point. Alfred Cowles, to whom this book is dedicated has indeed been honored.

Martin Beckmann

(Continued on p. 244)

VERTEX EXCITED SURFACE WAVES ON ONE FACE OF A RIGHT ANGLED WEDGE*

BY

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Abstract. The problem of the propagation of electromagnetic waves by a magnetic line dipole source located at the corner of a right angled wedge is considered. It is assumed that an impedance or mixed boundary condition is prescribed on one of the wedge surfaces and that a homogeneous boundary condition is prescribed on the other. The impedance boundary condition is such that surface waves are generated. The amplitude of the surface wave generated is determined. A comparison is made between the magnitude of the surface wave for this problem and that of a magnetic-line dipole source located at the corner of a right angled wedge with the same impedance boundary condition prescribed on both* surfaces. The far field amplitude of the radiated electromagnetic field is also given as an elementary function of the angle of observation.

1. Introduction. The problem of the propagation of electromagnetic waves produced by a magnetic line dipole source located at the corner of a right angled wedge is considered. It is assumed that an impedance or mixed boundary condition is prescribed on one of the wedge surfaces and that a homogeneous boundary condition is prescribed on the other. The impedance boundary condition is such that surface waves are generated. The amplitude of the surface wave is determined. A comparison is made between the magnitude of the surface wave for this problem and that of a magnetic-line dipole source located at the corner of a right angled wedge with the same impedance boundary condition prescribed on *both* surfaces. The latter problem has already been treated by the authors (see Karp and Karal [3]). The comparison of the surface waves for the two different configurations is made on the assumption that the sources have the same strength. The far field representation of the radiated electromagnetic field is also given.

The impedance boundary condition prescribed in the present problem is of the form

$$\frac{1}{r}\frac{\partial u}{\partial \theta} - \lambda u = 0 \qquad \theta = \alpha \tag{1.1}$$

where u is the z component of the magnetic vector, r and θ are the usual polar coordinates, α is the wedge angle, and λ is a constant characteristic of the surface. The value of λ is given by

$$\lambda = +i\omega\epsilon Z = +i\omega\epsilon(R - iX) \tag{1.2}$$

where ϵ is the permittivity of free space, ω is the angular frequency and Z, R and X are the impedance, resistance and reactance of the surface, respectively. It is well known that for homogeneous media of large finite conductivity, R and X are positive in sign, small in magnitude and approximately equal. For corrugated or dielectric-coated surfaces,

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*See Ref. [3] below.

R and X are positive in sign and R is much smaller than X. Hence X is positive in either case and Eq. (1.2) implies that

$$\mathrm{Re}\,\lambda > 0. \tag{1.3}$$

For illustrations of these boundary conditions see the Introduction of Refs. [3] and [7] where the appropriate references are cited. The condition $R \ll X$ is of course the most important special case insofar as surface waves are concerned.

The problem we treat is not separable because of the mixed boundary conditions. This difficulty can be overcome by the introduction of an auxiliary function which is a linear combination of the magnetic-field and its cartesian derivatives. The auxiliary function is chosen in such a way that it satisfies the wave equation and simple homogeneous boundary conditions. Once the auxiliary function is obtained, the original field can be determined by solving a partial differential equation. This idea is due to Stoker [6] and Lewy [4] who studied problems in water wave theory. Similar ideas have been employed by the authors in the solution of diffraction problems arising in electromagnetic theory. See [1], [2], [3] and [7].

Section 2 contains the exact solution of the problem stated in the preceding paragraphs. The limiting cases of small and large λ/k are studied and simplified expressions for the surface waves in both cases are given. A comparison between these results and those obtained in the problem of a line source located at the tip of a right angled wedge with the same impedance boundary condition prescribed on both surfaces is made. In Sec. 3 we discuss the radiated far field and obtain simple expressions for arbitrary values of λ.

2. Solution. Consider the right angled wedge $y = 0$, $x > 0$ and $x = 0$, $y < 0$, and suppose there is a magnetic-line dipole located at the origin $x = 0$, $y = 0$. (See Fig. 1.) The boundary conditions on the surface are given by

$$\begin{cases} \dfrac{\partial u}{\partial y} = 0, & y = 0, \quad x > 0 \\[2ex] \dfrac{\partial u}{\partial x} - \lambda u = 0, & x = 0, \quad y < 0 \end{cases} \tag{2.1}$$

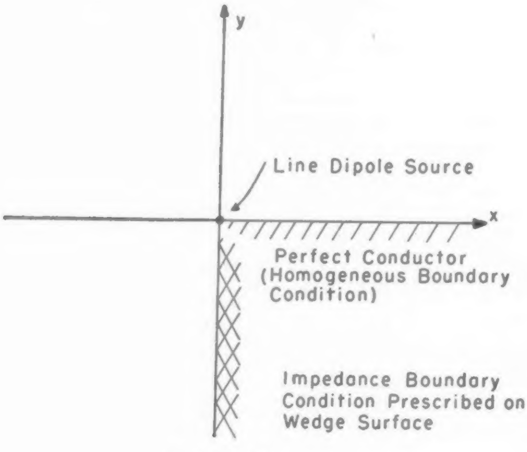

FIGURE I

where u is the magnetic vector H_z and λ is a constant related to the material comprising the surface. We wish to solve the time reduced Maxwell's equations, subject to the prescribed conditions, and obtain the amplitude of the surface wave propagated on the surface $x = 0, y < 0$.

The time dependent form of Maxwell's equations is

$$\begin{cases} \text{curl } \mathbf{H} = -i\omega\epsilon\mathbf{E}, \\ \text{curl } \mathbf{E} = i\omega u\mathbf{H}, \end{cases} \tag{2.2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field intensities, and ϵ and u are the permittivity and magnetic permeability of free space. We assume the time dependence to be of the form $\exp(-i\omega t)$. Because of the geometry, the field produced is independent of z and hence the field is completely determined by the value of H_z. We have

$$H_x = H_y = E_z = 0 \tag{2.3}$$

and

$$\begin{bmatrix} E_x = -\dfrac{1}{i\omega\epsilon}\dfrac{\partial H_z}{\partial y}, \\ E_y = +\dfrac{1}{i\omega\epsilon}\dfrac{\partial H_z}{\partial x}. \end{bmatrix} \tag{2.4}$$

The field component $H_z \equiv u$ satisfies the equation

$$(\nabla^2 + k^2)u = -4\pi\,\delta(x - 0)\,\delta(y - 0), \tag{2.5}$$

where ∇^2 is the rectangular Laplacian, k is the propagation constant of free space and δ is the Dirac delta function. Therefore, the mathematical problem reduces to that of solving the inhomogeneous wave equation (2.5) subject to the boundary conditions given by (2.1). In addition to the prescribed boundary conditions, we require that the far field be outgoing and that the total magnetic field, excluding the source, be finite everywhere.

Let us make the substitution

$$v = \frac{\partial u}{\partial x} - \lambda u. \tag{2.6}$$

Then v satisfies the following equation:

$$(\nabla^2 + k^2)v = 0, \qquad x^2 + y^2 \neq 0, \qquad 0 \leq \theta \leq \frac{3\pi}{2}, \tag{2.7}$$

subject to the simple conditions

$$\begin{cases} (1)\ \dfrac{\partial v}{\partial y} = 0 \qquad y = 0, \qquad x > 0 \\ (2)\ v = 0 \qquad x = 0, \qquad y < 0 \\ (3)\ \text{Outgoing waves at infinity.} \end{cases} \tag{2.8}$$

If the problem for v can be solved, then we can obtain the solution for the problem involving u since (2.6) can be integrated.

A particular solution of (2.6) is

$$u_p(x, y) = -\exp(\lambda\xi) \int_x^\infty \exp(-\lambda x) v(\xi, y)\, d\xi. \tag{2.9}$$

The range of integration is dictated by the condition that Re $\lambda > 0$. It is important to point out that the function v does not have to be finite at the origin. Thus we should first introduce all radiating solutions, no matter how singular at the origin, that satisfy the appropriate boundary conditions for v. Out of this class of solutions we finally select only those that yield an everywhere finite value of the total magnetic-field (excluding the source) in the original physical problem involving u. By employing the method of separation of variables, we find that the most general representation for v which satisfies conditions (2.8) is of the form $\sum a_n H^{(1)}_{(2n+1)/3}(kr) \cos[(2n+1)/3]\theta$, where n is an integer and $H^{(1)}_{(2n+1)/3}(kr)$ is a Hankel function of the first kind of order $(2n+1)/3$. Since v cannot be too singular, the only admissible functions of this class are

$$v(x, y) = c_0 H_1^{(1)}(kr) \cos\theta + c_1 H_{1/3}^{(1)}(kr) \cos\frac{\theta}{3}. \tag{2.10}$$

The above expression for v is a wave function, defined, by means of (2.7), in the angular sector $0 \leq \theta \leq 3\pi/2$. We note, however, that we can extend the range of definition of v to include the angular sector $0 \leq \theta \leq 2\pi$ by means of the reflection principle since v vanishes on the negative y-axis. This extension is necessary since the range of integration of the particular solution (2.9) extends into the fourth quadrant for negative values of y. Using this extended definition of v, the particular solution (2.9) becomes

$$u_p(x, y) = -\exp(\lambda\xi) \int_x^\infty \exp(-\lambda\xi)\left\{c_0 H_1^{(1)}(kr)\cos\theta + c_1 H_{1/3}^{(1)}(kr)\cos\frac{\theta}{3}\right\} d\xi. \tag{2.11}$$

The above result, though well defined, does not satisfy all the conditions of the problem since it is not regular across the line $y = 0$, $x < 0$. This difficulty can be overcome by adding a complementary solution, u_c say, of the inhomogeneous equation (2.6) to (2.11). We then obtain the representation

$$\begin{aligned} u(x, y) = -\exp(\lambda x) \int_x^\infty \exp(-\lambda\xi)\left\{c_0 H_1^{(1)}(kr)\cos\theta + c_1 H_{1/3}^{(1)}(kr)\cos\frac{\theta}{3}\right\} d\xi \\ + \begin{bmatrix} c_2 \exp[\lambda x - i(k^2 + \lambda^2)^{1/2} y], & y < 0 \\ 0 \quad\quad\quad\quad , & y > 0 \end{bmatrix}, \end{aligned} \tag{2.12}$$

where $r = (\xi^2 + y^2)^{1/2}$, $\theta = \arctan \xi/y$ and $0 \leq \theta \leq 2\pi$. The complementary solution u_c, which is indicated in square brackets in (2.12), is obviously not a wave function since it is discontinuous across the line $y = 0$. However, although each solution separately is discontinuous together with its y-derivative along the negative x-axis, the solution $u = u_p + u_c$ of (2.12) must be regular there. From this fact we can derive two relations between the constants c_0, c_1 and c_2 by calculating the discontinuities of u and $\partial u/\partial y$ across the negative x-axis and setting the discontinuities equal to zero. These calculations completely solve the problem because the constant c_0 is related to the source strength and is known. Hence there are a sufficient number of equations to determine all of the unknown constants. We note that the form of the term in brackets in (2.12) is that of a surface wave with an amplitude given by c_2.

Before proceeding to the calculation of the discontinuities and the value of the

constants c_1 and c_2, we first simplify the expression for u and then determine $\partial u/\partial y$. By using the relation

$$\frac{\partial}{\partial x} H_0^{(1)}(kr) = -kH_1^{(1)}(kr) \cos \theta \tag{2.13}$$

and integrating by parts, (2.12) becomes

$$\begin{aligned} u(x, y) = &-\frac{c_0}{k}\left\{H_0^{(1)}(kr) - \lambda \exp(\lambda x) \int_x^\infty \exp(-\lambda\xi) H_0^{(1)}(kr)\, d\xi\right\} \\ &- c_1 \exp(\lambda x) \int_x^\infty \exp(-\lambda\xi) H_{1/3}^{(1)}(kr) \cos\frac{\theta}{3}\, d\xi \\ &+ \begin{bmatrix} c_2 \exp[\lambda x - i(k^2+\lambda^2)^{1/2} y], & y < 0 \\ 0 \qquad, & y > 0 \end{bmatrix}. \end{aligned} \tag{2.14}$$

At this stage we can determine c_0 since we recognize the Hankel function $H_0^{(1)}(kr)$ as being the Green's function for a line source. The solution of (2.5) for a simple point source located at the origin is given by the well-known result

$$G(r, \theta; 0, 0) = i\pi H_0^{(1)}(kr). \tag{2.15}$$

Hence the proper normalization for a point source is obtained provided we let

$$c_0 = -i\pi k. \tag{2.16}$$

We next determine $\partial u/\partial y$. We have

$$\begin{aligned} u_y(x, y) \equiv \frac{\partial u}{\partial y}(x, y) = &-\frac{c_0}{k}\left[\frac{\partial}{\partial y} H_0^{(1)}(kr) - \lambda \exp(\lambda x) \int_x^\infty \exp(-\lambda\xi) \frac{\partial}{\partial y}\{H_0^{(1)}(kr)\}\, d\xi\right] \\ &- c_1 \exp(\lambda x) \int_x^\infty \exp(-\lambda\xi) \frac{\partial}{\partial y}\left\{H_{1/3}^{(1)}(kr) \cos\frac{\theta}{3}\right\} d\xi \\ &+ \begin{bmatrix} -c_2 i(k^2+\lambda^2)^{1/2} \exp[\lambda x - i(k^2+\lambda^2)^{1/2} y], & y < 0 \\ 0 \qquad, & y > 0 \end{bmatrix}, \end{aligned} \tag{2.17}$$

where we have taken the derivative inside the integral sign. By using the following relations

$$\frac{\partial}{\partial y} H_0^{(1)}(kr) = -kH_1^{(1)}(kr) \sin \theta, \tag{2.18}$$

$$\frac{\partial}{\partial y}\left\{H_{1/3}^{(1)}(kr) \cos\frac{\theta}{3}\right\} - \frac{\partial}{\partial x}\left\{H_{1/3}^{(1)}(kr) \sin\frac{\theta}{3}\right\} = k \exp[i(2\pi/3)] H_{2/3}^{(1)}(kr) \sin\frac{2\theta}{3} \tag{2.19}$$

and integrating by parts we obtain

$$\begin{aligned} u_y(x, y) = &-\frac{c_0}{k}\left\{-kH_1^{(1)}(kr) \sin\theta + k\lambda \exp(\lambda x) \int_x^\infty \exp(-\lambda\xi) H_1^{(1)}(kr) \sin\theta\, d\xi\right\} \\ &+ c_1 H_{1/3}^{(1)}(kr) \sin\frac{\theta}{3} - c_1\lambda \exp(\lambda x) \int_x^\infty \exp(-\lambda\xi) H_{1/3}^{(1)}(kr) \sin\frac{\theta}{3}\, d\xi \\ &- c_1 k \exp[i(2\pi/3)] \exp(\lambda x) \int_x^\infty \exp(-\lambda\xi) H_{2/3}^{(1)}(kr) \sin\frac{2\theta}{3}\, d\xi \\ &+ \begin{bmatrix} -c_2 i(k^2+\lambda^2)^{1/2} \exp[\lambda x - i(k^2+\lambda^2)^{1/2} y], & y < 0 \\ 0 \qquad, & y > 0 \end{bmatrix}. \end{aligned} \tag{2.20}$$

It was noted earlier that the constants c_1 and c_2 can be determined by requiring that u and its derivatives be continuous throughout the region of physical space $(0 \leq \theta \leq 3\pi/2)$. This is equivalent to calculating the discontinuities of u and $\partial u/\partial y$ across the negative x-axis and setting the discontinuities equal to zero. Hence for $u(x, y)$ to be a wave function, we require that the following jump conditions be satisfied:

$$\begin{cases} (\mathrm{I})[u] \equiv \{u(x, 0^+) - u(x, 0^-)\} = 0 \ , & x < 0 \\ (\mathrm{II})[u_y] \equiv \{u_y(x, 0^+) - u_y(x, 0^-)\} = 0, & x < 0. \end{cases} \tag{2.21}$$

The first jump condition yields immediately

$$c_2 = -\tfrac{3}{2}c_1 I_{1/3}(k, \lambda), \tag{2.22}$$

where we have used the notation

$$I_\nu(k, \lambda) = \int_0^\infty \exp(-\lambda\xi) H_\nu^{(1)}(k \mid \xi \mid)\, d\xi. \tag{2.23}$$

We note that

$$I_\nu(k, \lambda) = \frac{i}{\sin \nu\pi} \frac{\exp[-i\nu(\pi/2)]}{k^\nu(\lambda^2 + k^2)^{1/2}} \cdot \{\exp[-i\nu(\pi/2)]([\lambda^2 + k^2]^{1/2} - \lambda)^\nu - \exp[i\nu(\pi/2)]([\lambda^2 + k^2]^{1/2} + \lambda)^\nu\}, \tag{2.24}$$

where $-1 > \operatorname{Re} \nu < +1$. (See Magnus and Oberhettinger [5].) The second jump condition requires more careful treatment. If we substitute (2.20) into jump condition (II) and use (2.16) we obtain

$$\begin{aligned}
[u_y] = \lim_{y \to +0} 2\pi i k\lambda \exp(\lambda x) \int_x^\infty \exp(-\lambda\xi) H_1^{(1)}(k[\xi^2 + y^2]^{1/2}) \frac{y}{(\xi^2 + y^2)^{1/2}}\, d\xi \\
+ c_1\lambda \exp(\lambda x) \sin\frac{2\pi}{3} \int_0^\infty \exp(-\lambda\xi) H_{1/3}^{(1)}(k \mid \xi \mid)\, d\xi \\
+ c_1 k \exp[i(2\pi/3)] \exp(\lambda x) \sin\frac{4\pi}{3} \int_0^\infty \exp(-\lambda\xi) H_{2/3}^{(1)}(k \mid \xi \mid)\, d\xi \\
+ c_2 i(k^2 + \lambda^2)^{1/2} \exp(\lambda x) = 0.
\end{aligned} \tag{2.25}$$

Let us consider the expression

$$J = \lim_{y \to +0} \int_x^\infty \exp(-\lambda\xi) H_1^{(1)}(k[\xi^2 + y^2]^{1/2}) \frac{y}{(\xi^2 + y^2)^{1/2}}\, d\xi. \tag{2.26}$$

The integrand in (2.26) tends to zero with y except in the neighborhood of $\xi = 0$. Hence the only possible contribution occurs when ξ lies in an interval $-\epsilon$ to ϵ, where ϵ is arbitrarily small. Since we assume that y is already small, we can use the asymptotic representations for the exponential and Hankel functions appearing in the integrand. Hence

$$J = \frac{-2i}{\pi k} \lim_{y \to +0} \int_{-\epsilon}^{\epsilon} \frac{y}{\xi^2 + y^2}\, d\xi. \tag{2.27}$$

If we make the substitution $\xi = yt$, the above integral becomes

$$J = -\frac{2i}{\pi k} \int_{-\infty}^\infty \frac{dt}{1 + t^2} = -2\,\frac{i}{k}. \tag{2.28}$$

Substituting this result into (2.25) and using the notation defined in (2.23), the second jump condition becomes

$$+ 4\pi\lambda + c_1\lambda \sin\frac{2\pi}{3} I_{1/3}(k, \lambda)$$
$$+ c_1 k \exp[i(2\pi/3)] \sin\frac{4\pi}{3} I_{2/3}(k, \lambda) + c_2 i(k^2 + \lambda^2)^{1/2} = 0. \tag{2.29}$$

By substituting (2.22) into (2.29) we find c_1. We obtain

$$c_1 = -\frac{4\pi}{3^{1/2}} \frac{2\lambda}{(\lambda - i3^{1/3}[k^2 + \lambda^2]^{1/2})I_{1/3} - k \exp[i(2\pi/3)]I_{2/3}} \tag{2.30}$$

and

$$c_2 = +2\,3^{1/2}\pi \frac{2\lambda I_{1/3}}{(\lambda - 3^{1/2}i[k^2 + \lambda^2]^{1/2})I_{1/3} - k \exp[i(2\pi/3)]I_{2/3}}. \tag{2.31}$$

The value of c_2 is the amplitude of the surface wave propagated along the wedge surface.

We next determine the values of c_1 and c_2 for small and large values of λ/k. When λ/k is small it is easily shown that

$$I_{1/3} \approx \frac{1}{k}(1 - i3^{-1/2}) = \frac{2}{k} 3^{-1/2} \exp[-i(\pi/6)] \tag{2.32}$$

$$I_{2/3} \approx \frac{1}{k}(1 - i3^{1/2}) = \frac{2}{k} \exp[-i(\pi/3)] \tag{2.33}$$

and the approximate expressions for c_1 and c_2 become

$$c_1 \approx \frac{2\pi\lambda}{3^{1/2}} \exp[-i(\pi/3)] \tag{2.34}$$

$$c_2 \approx 2\pi i\left(\frac{\lambda}{k}\right). \tag{2.35}$$

When λ/k is large we find that

$$I_{1/3} \approx -i\frac{2}{3^{1/2}}\left(\frac{2}{k}\right)^{1/3} \frac{1}{\lambda^{2/3}} \tag{2.36}$$

$$I_{2/3} \approx -i\frac{2}{3^{1/2}}\left(\frac{2}{k}\right)^{2/3} \frac{1}{\lambda^{1/3}} \tag{2.37}$$

and the approximate expressions for c_1 and c_2 become

$$c_1 \approx \pi k \exp[-i(\pi/6)]\left(\frac{2\lambda}{k}\right)^{2/3} \tag{2.38}$$

$$c_2 \approx 2\,3^{1/2}\pi \exp[i(\pi/3)]. \tag{2.39}$$

It is of interest to compare the amplitude of the surface wave generated in this problem with that generated by a line source located at the tip of a right-angled wedge with the same impedance boundary condition prescribed on *both* surfaces. The latter problem has previously been solved [3] and the amplitude of the surface wave propagated on the wedge surface $x = 0$, $y < 0$ is given by

$$c_3 = 4\,3^{1/2}\pi \frac{\lambda I_{2/3} - k \exp [i(\pi/3)] I_{1/3}}{\{[i(k^2 + \lambda^2)^{1/2} - \lambda 3^{1/2}] I_{2/3} + 3^{1/2} k \exp [i(\pi/3)] I_{1/3}\}} \cdot \frac{\lambda}{[i(k^2 + \lambda^2)^{1/2} - \lambda]}. \tag{2.40}$$

When λ/k is small we find that

$$c_3 \approx 2\pi i \left(\frac{\lambda}{k}\right) \tag{2.41}$$

and when λ/k is large

$$c_3 \approx \frac{4\pi}{(1 - i)(1 - i3^{1/2})} = 2\pi \left(\frac{3}{2}\right)^{1/2} \exp [i(5\pi/12)]. \tag{2.42}$$

Hence, when the value of λ/k is small, the ratio of the amplitude of the surface wave on a wedge with an impedance boundary condition on one surface to that on both surfaces is given by unity. When the value of λ/k is large, the ratio is given by $2^{1/2}$. In the limit $\lambda/k \to \infty$ therefore, the total energy that goes into surface waves in our present configuration is the same as the energy so utilized when both faces of the wedge can support surface waves.

3. Determination of the radiated far field. We have already obtained the exact solution for a line source located at the tip of a right-angled wedge with an impedance type boundary condition prescribed on one surface. The exact solution as given by (2.14), however, is complicated and it is desirable to find a simple expression for the radiated far field. From (2.6) and (2.10) we have

$$v = \frac{\partial u}{\partial x} - \lambda u \tag{3.1}$$

and

$$v = c_0 H_1^{(1)}(kr) \cos \theta + c_1 H_{1/3}^{(1)}(kr) \cos \frac{\theta}{3}. \tag{3.2}$$

When kr is large (3.2) becomes

$$v = c_0 \left(\frac{2}{\pi kr}\right)^{1/2} \exp \{i[kr - (3\pi/4)]\} \cos \theta + c_1 \left(\frac{2}{\pi kr}\right)^{1/2} \exp \{i[kr - (5\pi/12)]\} \cos \frac{\theta}{3}. \tag{3.3}$$

We expect the far field of the z-component of the magnetic field vector to have a similar form, except that it contains surface wave terms which are appreciable near the wedge surface $x = 0$, $y < 0$. Hence for a fixed θ, as $r \to \infty$ we have

$$u = \left[\frac{l(\theta)}{r^{1/2}} \exp (ikr) + 0\left(\frac{1}{r^{3/2}}\right)\right], \qquad \theta \neq 3\pi/2, \tag{3.4}$$

where $l(\theta)$ is unknown. Now

$$\frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta}. \tag{3.5}$$

If we substitute (3.3), (3.4) and (3.5) into (3.1), and keep terms of order $r^{-1/2}$, we obtain

$$l(\theta) = \left(\frac{2}{\pi k}\right)^{1/2} \frac{c_0 \exp\,[-i(3\pi/4)] \cos\theta + c_1 \exp\,[-i(5\pi/12)] \cos\dfrac{\theta}{3}}{ik\cos\theta - \lambda}. \tag{3.6}$$

Hence the desired radiated field is given by

$$u(r,\theta) = \left(\frac{2}{\pi k r}\right)^{1/2} e^{ikr}\, \frac{c_0 \exp\,[-i(3\pi/4)] \cos\theta + c_1 \exp\,[-i(5\pi/12)] \cos\dfrac{\theta}{3}}{ik\cos\theta - \lambda} + 0\left(\frac{1}{r^{3/2}}\right) \tag{3.7}$$

where c_0, c_1 and c_2 are given by (2.16), (2.30) and (2.31), respectively.

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BOOK REVIEWS

(*Continued from p. 234*)

Applications of the theory of matrices. By F. R. Gantmacher. Interscience Publishers, New York, London, 1959. ix + 317. $9.00.

This is a translation of the second part of the author's *A Theory of Matrices* which was published in 1954 in Russian. As such it constitutes a unique mathematical monograph. The applications alluded to in the title lie in other fields of mathematics. The reader should possess considerable maturity in these fields as well as in matrix theory in order to appreciate the elegance of the author's developments.

There are five chapters. The first two, treating the normal forms of complex matrices and singular pencils of matrices, are quite formal being little more than a sequence of lemmas and theorems. The third chapter treating matrices with non negative elements begins formally but becomes more readable when it turns to applications to Markov chains with a finite number of states and to oscillating matrices. The two final chapters contain the major applications. Chapter 4 is a study of systems of linear differential equations and Chapter 5 is devoted to the Routh-Hurwitz problem.

As an indication of the level, the chapter on differential equations includes Lyapunov's stability theory, the theory of singular points and works up to conclude that certain results of V. Volterra and G. D. Birkhoff concerning isolated singular points were erroneous due to insufficient examination of the degenerate cases involving nonlinear divisors.

The author's developments are elegant and worthy of study by experts in the fields mentioned. In one case they appeared overly elaborate to this reviewer. In Chapter 3 the properties of oscillating matrices are developed to show that certain vibratory systems (e.g., a stretched string carrying n masses) possess the Sturm properties. Apparently the author was unaware that in 1884 E. J. Routh had demonstrated these same results in a much more direct manner by extending Sturm's argument to finite difference equations.

Stephen H. Crandall

Matrix calculus. By. E. Bodewig, Second Edition. North Holland Publishing Co., Amsterdam, and Interscience Publishers, Inc., New York, 1959. xi + 452 pp. $9.50.

Recent progress in digital computer techniques for the determination of the eigenvalues and -vectors of large matrices is reflected in this new edition by the inclusion of Lanczos's *pq*-algorithm, Rutishauser's *LR*-algorithm and Wilkinson's method. Several hitherto unpublished results by the author are also described. The work retains its character as an invaluable fund of information on matrix methods and their relative merits in different context.

Walter Freiberger

Mathematical methods and theory in games, programming, and economics. By Samuel Karlin. Volumes I and II. Addison-Wesley Publishing Co., Inc., Reading, Mass., and London, 1959. x + 433 pp. (Vol. I), xi + 386 pp. (Vol. II). $12.50 each.

The last twenty years have seen the remarkable growth of a new branch of applied mathematics—the use of mathematics in decision making within both business and government. These two books present a systematic and elegant account of many of the mathematical methods which have been developed for this subject, written by one of its leading practitioners.

The first four chapters of Volume I are fairly standard material about matrix games. Then come linear programming—theory and computational methods—and a chapter on nonlinear programming. The various famous special types of problems (optimal assignment, transportation, network flow, etc.) are taken up in detail. Fenchel's theory of conjugate convex functions is developed from a new geometric viewpoint, and is applied to get a duality theorem for nonlinear programming. The rest of Volume I deals with theories of production and consumption, with welfare economics, and with the dynamics

(*Continued on p. 270*)

A GENERAL THEOREM CONCERNING THE STABILITY OF A PARTICULAR NON-NEWTONIAN FLUID*

BY

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Summary. It is the intention of the present paper to prove a theorem concerning the stability of a particular non-Newtonian fluid suggested to the author by Professor R. S. Rivlin of Brown University. The method used in proving this theorem is similar to that employed by H. Schlichting in his proof of a similar theorem for an inviscid fluid which was originally established by Lord Rayleigh. The acceleration gradients introduced by the non-Newtonian fluid model into the constitutive equations are found to alter the stability criterion set forth by Rayleigh for an inviscid fluid.

1. Introduction. As early as 1880 Rayleigh [1] proved that for an inviscid fluid the existence of a point of inflection in the velocity profile of a steady one-dimensional basic flow is a necessary condition for the growth of a superimposed two-dimensional disturbance. It is the intention of the present paper to prove a similar theorem for a particular non-Newtonian fluid suggested to the author by Professor R. S. Rivlin.* The method used in proving this theorem is similar to that employed by Schlichting [2] in his proof of the Rayleigh theorem.

2. The constitutive equations and equations of motion. Let $X_j (j = 1, 2, 3)$ be the coordinates referred to a rectangular Cartesian coordinate system x_i, of a generic particle of a continuous medium in the undeformed state at time t_0. Let x_i be the coordinates of the same particle in the deformed state at time t. It then follows that the components of velocity v_i and the components of acceleration a_i of the particle are given by

$$v_i = \frac{\partial x_i}{\partial t} \quad \text{and} \quad a_i = \frac{\partial^2 x_i}{\partial t^2}, \tag{2.1}$$

where x_i are regarded as single-value continuous functions of $X_k (k = 1, 2, 3)$ and t, having as many continuous derivatives as the analysis requires. As is well known, if v_i are considered to be functions of x_k and t, then

$$a_i = \frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}. \tag{2.2}$$

Now the equations of motion are

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}\right) + \rho f_i = \frac{\partial t_{il}}{\partial x_l}, \qquad (l = 1, 2, 3), \tag{2.3}$$

where ρ is the mass of the medium per unit volume and f_i are the components in the coordinate directions of the body force per unit mass, also measured in the deformed state. The components of stress t_{il} resulting from the deformation are defined as follows; t_{i1}, t_{i2} and t_{i3} are the components of the force per unit area in the positive direction of the x_1, x_2 and x_3 axes respectively, measured in the deformed state, exerted across an

[1]Received October 28, 1958; revised manuscript received August 12, 1959.

element of area at (x_1, x_2, x_3) normal to the x_j axis, by the material on the positive side of the element upon the material on the negative side of the element.

Rivlin [3] showed that if t_{jl} at the point x_k and at time t are assumed to be polynomials in the velocity gradients $\partial v_m/\partial x_n$ $(m, n = 1, 2, 3)$ and the acceleration gradients $\partial a_m/\partial x_n$ and if, in addition, the medium is assumed to be isotropic at time t_0, then the stress matrix $\mathbf{T} = \| t_{jl} \|$ is expressible in the form

$$\begin{aligned}\mathbf{T} = \varphi_0\mathbf{I} + \varphi_1\mathbf{A}_1 + \varphi_2\mathbf{A}_2 + \varphi_3\mathbf{A}_1^2 + \varphi_4\mathbf{A}_2^2 + \varphi_5(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) \\ + \varphi_6(\mathbf{A}_1^2\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1^2) + \varphi_7(\mathbf{A}_1\mathbf{A}_2^2 + \mathbf{A}_2^2\mathbf{A}_1) + \varphi_8(\mathbf{A}_1^2\mathbf{A}_2^2 + \mathbf{A}_2^2\mathbf{A}_1^2),\end{aligned} \tag{2.4}$$

where $\mathbf{I}$ is the unit matrix, $\mathbf{A}_1$ and $\mathbf{A}_2$ are symmetric kinematic matrices defined by

$$\mathbf{A}_1 = \left\| \frac{\partial v_j}{\partial x_l} + \frac{\partial v_l}{\partial x_j} \right\| \quad \text{and} \quad \mathbf{A}_2 = \left\| \frac{\partial a_j}{\partial x_l} + \frac{\partial a_l}{\partial x_j} + 2\frac{\partial v_m}{\partial x_j}\frac{\partial v_m}{\partial x_l} \right\| \tag{2.5}$$

and φ_q $(q = 0, 1, 2, \cdots, 8)$ are polynomials in $tr\mathbf{A}_1$, $tr\mathbf{A}_2$, $tr\mathbf{A}_1^2$, $tr\mathbf{A}_2^2$, $tr\mathbf{A}_1^3$, $tr\mathbf{A}_2^3$, $tr\mathbf{A}_1\mathbf{A}_2$, $tr\mathbf{A}_1^2\mathbf{A}_2$, $tr\mathbf{A}_1\mathbf{A}_2^2$ and $tr\mathbf{A}_1^2\mathbf{A}_2^2$. In a later paper Rivlin [4] pointed out that for an incompressible material, the stress corresponding to a specific state of flow is indeterminate to the extent of an arbitrary hydrostatic pressure p. Since in the present paper we shall confine our analysis to an incompressible fluid, we may replace φ_0 in equation (2.4) by $-p$. We shall also restrict our investigation to a fluid for which φ_1 and φ_2 are constants and φ_q $(q = 3, 4, 5, \cdots, 8)$ are identically zero.

Equation (2.4) now takes the form

$$\mathbf{T} = -p\mathbf{I} + \varphi_1\mathbf{A}_1 + \varphi_2\mathbf{A}_2, \tag{2.6}$$

or alternatively

$$t_{jl} = -p\,\delta_{jl} + \varphi_1\left(\frac{\partial v_j}{\partial x_l} + \frac{\partial v_l}{\partial x_j}\right) + \varphi_2\left(\frac{\partial a_j}{\partial x_l} + \frac{\partial a_l}{\partial x_j} + 2\frac{\partial v_m}{\partial x_j}\frac{\partial v_m}{\partial x_l}\right). \tag{2.7}$$

Introducing Eqs. (2.2) into Eqs. (2.7), we obtain

$$\begin{aligned} t_{jl} = -p\,\delta_{jl} + \varphi_1\left(\frac{\partial v_j}{\partial x_l} + \frac{\partial v_l}{\partial x_j}\right) + \varphi_2\left(\frac{\partial^2 v_j}{\partial t\,\partial x_l} + \frac{\partial v_m}{\partial x_l}\frac{\partial v_j}{\partial x_m}\right. \\ \left. + v_m\frac{\partial^2 v_j}{\partial x_l\,\partial x_m} + \frac{\partial^2 v_l}{\partial t\,\partial x_j} + \frac{\partial v_m}{\partial x_j}\frac{\partial v_l}{\partial x_m} + v_m\frac{\partial^2 v_l}{\partial x_j\,\partial x_m} + 2\frac{\partial v_m}{\partial x_j}\frac{\partial v_m}{\partial x_l}\right).\end{aligned} \tag{2.8}$$

Since we have assumed that the fluid is incompressible, the continuity equation is

$$\frac{\partial v_m}{\partial x_m} = 0. \tag{2.9}$$

Introducing the constitutive equations (2.8) in the equations of motion (2.3), employing the incompressibility condition (2.9) and neglecting body forces, we obtain

$$\begin{aligned}\rho\left(\frac{\partial v_j}{\partial t} + v_k\frac{\partial v_j}{\partial x_k}\right) = -\frac{\partial p}{\partial x_j} + \varphi_1\frac{\partial^2 v_j}{\partial x_l\,\partial x_l} + \varphi_2\left(\frac{\partial^3 v_j}{\partial t\,\partial x_l\,\partial x_l}\right. \\ + \frac{\partial v_j}{\partial x_m}\frac{\partial^2 v_m}{\partial x_l\,\partial x_l} + 2\frac{\partial v_m}{\partial x_l}\frac{\partial^2 v_j}{\partial x_l\,\partial x_m} \\ + v_m\frac{\partial^3 v_j}{\partial x_l\,\partial x_l\,\partial x_m} + 2\frac{\partial v_m}{\partial x_l}\frac{\partial^2 v_l}{\partial x_j\,\partial x_m} \\ \left. + 2\frac{\partial v_m}{\partial x_l}\frac{\partial^2 v_m}{\partial x_j\,\partial x_l} + 2\frac{\partial v_m}{\partial x_j}\frac{\partial^2 v_m}{\partial x_l\,\partial x_l}\right).\end{aligned} \tag{2.10}$$

3. Development of the stability equation. Consider next a two-dimensional steady laminar flow with velocity components

$$W_1 = W_1(x_2) \quad \text{and} \quad W_2 = W_3 = 0. \tag{3.1}$$

Examples of such flows are the flow between a pair of parallel plates sufficiently removed from the intake section, and the flow in the boundary layer along a flat plate excluding the region of its leading edge. The pressure P necessary to maintain such flows is in general a function of both x_1 and x_2.

We shall superimpose upon the laminar flow a small two dimensional disturbance with velocity components and associated pressure given by

$$u_1 = u_1(x_1, x_2, t), \qquad u_2 = u_2(x_1, x_2, t), \qquad u_3 = 0 \quad \text{and} \quad p^* = p^*(x_1, x_2, t). \tag{3.2}$$

Thus the velocity components and pressure of the composite flow are

$$v_1 = W_1 + u_1, \qquad v_2 = u_2, \qquad v_3 = 0 \quad \text{and} \quad p = P + p^*. \tag{3.3}$$

Further, we shall require that the composite flow satisfy the same boundary conditions as the steady laminar flow. Thus the disturbance satisfies the boundary conditions

$$u_1 = u_2 = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 = L \tag{3.4}$$

in the case of flow between parallel plates separated by a distance L, and

$$u_1 = u_2 = 0 \quad \text{at} \quad x_2 = 0 \quad \text{and} \quad x_2 \to \infty \tag{3.5}$$

in the case of boundary layer flow along a flat plate.

Next, introducing the velocity components and pressure given by (3.3) into Eqs. (2.9) and (2.10), and assuming that the velocity components u_i are sufficiently small so that terms of the second degree in u_i and derivatives of u_i may be neglected in comparison with terms of the first degree, we obtain

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \tag{3.6}$$

$$\begin{aligned}
\rho\left(\frac{\partial u_1}{\partial t} + W_1 \frac{\partial u_1}{\partial x_1} + W_1' u_2\right) = -\frac{\partial P}{\partial x_1} - \frac{\partial p^*}{\partial x_1} &+ \varphi_1(\nabla^2 u_1 + W_1'') \\
&+ \varphi_2\left(\nabla^2 \frac{\partial u_1}{\partial t} + 2W_1'' \frac{\partial u_2}{\partial x_2} + 4W_1' \frac{\partial^2 u_1}{\partial x_1 \, \partial x_2}\right. \\
&+ W_1' \nabla^2 u_2 + 3W_1'' \frac{\partial u_1}{\partial x_1} + W_1 \nabla^2 \frac{\partial u_1}{\partial x_1} \\
&\left. + W_1''' u_2 + 2W_1' \frac{\partial^2 u_2}{\partial x_1^2}\right)
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\rho\left(\frac{\partial u_2}{\partial t} + W_1 \frac{\partial u_2}{\partial x_1}\right) = -\frac{\partial P}{\partial x_2} - \frac{\partial p^*}{\partial x_2} + \varphi_1 \nabla^2 u_2 &+ \varphi_2\left(\nabla^2 \frac{\partial u_2}{\partial t}\right. \\
&+ 4W_1' \frac{\partial^2 u_2}{\partial x_1 \, \partial x_2} + 3W_1'' \frac{\partial u_2}{\partial x_1} + W_1 \nabla^2 \frac{\partial u_2}{\partial x_1} \\
&\left. + 4W_1' W_1'' + 4W_1'' \frac{\partial u_1}{\partial x_2} + 2W_1' \frac{\partial^2 u_1}{\partial x_2^2} + 2W_1' \nabla^2 u_1\right),
\end{aligned} \tag{3.8}$$

where $\nabla^2 = (\partial^2/\partial x_1^2) + (\partial^2/\partial x_2^2)$, and primes denote ordinary differentiation with respect to x_2.

Further, introducing Eq. (3.6) into Eqs. (3.7) and (3.8) and assuming the basic laminar flow satisfies the equations of motion, we obtain

$$\rho\left(\frac{\partial u_1}{\partial t} + W_1 \frac{\partial u_1}{\partial x_1} + W_1' u_2\right) = -\frac{\partial p^*}{\partial x_1} + \varphi_1 \nabla^2 u_1 + \varphi_2\left(\nabla^2 \frac{\partial u_1}{\partial t} + 3W_1' \frac{\partial^2 u_1}{\partial x_1\,\partial x_2} + W_1'' \frac{\partial u_1}{\partial x_1} + W_1 \nabla^2 \frac{\partial u_1}{\partial x_1} + W_1''' u_2 + 3W_1' \frac{\partial^2 u_2}{\partial x_1^2}\right) \tag{3.9}$$

and

$$\rho\left(\frac{\partial u_2}{\partial t} + W_1 \frac{\partial u_2}{\partial x_1}\right) = -\frac{\partial p^*}{\partial x_2} + \varphi_1 \nabla^2 u_2 + \varphi_2\left(\nabla^2 \frac{\partial u_2}{\partial t} + 2W_1' \frac{\partial^2 u_2}{\partial x_1\,\partial x_2} + 3W_1'' \frac{\partial u_2}{\partial x_1} + W_1 \nabla^2 \frac{\partial u_2}{\partial x_1} + 4W_1'' \frac{\partial u_1}{\partial x_2} + 4W_1' \frac{\partial^2 u_1}{\partial x_2^2}\right). \tag{3.10}$$

Differentiating Eq. (3.9) with respect to x_2 and Eq. (3.10) with respect to x_1, eliminating $\partial^2 p^*/\partial x_1 \partial x_2$ from the resulting equations and again employing Eq. (3.6), we arrive at

$$\left(\frac{\partial}{\partial t} + W_1 \frac{\partial}{\partial x_1} - \nu\nabla^2 - \gamma\nabla^2 \frac{\partial}{\partial t} - \gamma W_1 \nabla^2 \frac{\partial}{\partial x_1}\right)\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = (\gamma W'''' - W'')u_2, \tag{3.11}$$

where $\nu = \varphi_1/\rho$ and $\gamma = \varphi_2/\rho$.

Further, we shall assume that the velocity components of the disturbance are of the form

$$u_j(x_1, x_2, t) = u_j^*(x_2) \exp\,[iA(x_1 - Ct)], \qquad (j = 1, 2) \tag{3.12}$$

where $| u_j^* u_j^* |$ is the amplitude and A the wave number of the disturbance, and C is a complex number. The real part of C, C_r, is the phase velocity of the disturbance and the imaginary part, C_i, is the amplification factor. If $C_i > 0$, the disturbance tends to grow; if $C_i < 0$, the disturbance decays; and if $C_i = 0$, the disturbance is neutral.

Introducing the velocity components (3.12) into Eqs. (3.6) and (3.11), we obtain

$$iAu_1^* + u_2^{*\prime} = 0 \tag{3.13}$$

and

$$\begin{aligned} iA(W_1 - C)(u_1^{*\prime} - iAu_2^*) \\ - [\nu + iA\gamma(W_1 - C)](u_1^{*\prime\prime\prime} - A^2 u_1^{*\prime} - iAu_2^{*\prime\prime} + iA^3 u_2^*) \\ = (\gamma W_1'''' - W_1'')u_2^* . \end{aligned} \tag{3.14}$$

Replacing u_1^* in Eq. (3.14) by $iu_2^{*\prime}/A$ from Eq. (3.13), we obtain

$$\begin{aligned} iA(W_1 - C)(u_2^{*\prime\prime} - A^2 u_2^*) - [\nu + iA\gamma(W_1 - C)](u_2^{*\prime\prime\prime\prime} - 2A^2 u_2^{*\prime\prime} + A^4 u_2^*) \\ = iA(W_1'' - \gamma W_1'''')u_2^* . \end{aligned} \tag{3.15}$$

Writing Eq. (3.15) in dimensionless form by letting $\eta_2 = x_2/L$, $V_1 = W_1/W_0$, $w_2 = u_2^*/W_0$, $\alpha = AL$ and $c = C/W_0$ where L and W_0 are a characteristic length and a characteristic velocity respectively of the steady laminar flow, we arrive at

$$i\alpha(V_1 - c)(w_2'' - \alpha^2 w_2) - \left[\frac{1}{R} + \frac{i\alpha}{S}(V_1 - c)\right](w_2'''' - 2\alpha^2 w_2'' + \alpha^4 w_2) = i\alpha\left(V_1'' - \frac{1}{S} V_1''''\right)w_2 , \tag{3.16}$$

where $R = W_0 L/\nu$ is the Reynolds' number of the laminar flow, $S = L^2/\gamma$ is a non-dimensional parameter arising from the presence of non-Newtonian terms in the constitutive equations of the fluid, and primes denote ordinary differentiation with respect to η_2.

We see that if $S \to \infty$, Eq. (3.16) becomes the familiar Orr-Sommerfeld stability equation.

Finally, in terms of w_2 the boundary conditions (3.4) and (3.5) become

$$w_2 = w_2' = 0 \quad \text{at} \quad \eta_2 = 0 \quad \text{and} \quad \eta_2 = 1, \tag{3.17}$$

and

$$w_2 = w_2' = 0 \quad \text{at} \quad \eta_2 = 0 \quad \text{and} \quad \eta_2 \to \infty \tag{3.18}$$

respectively.

4. The general theorem. Under the assumption that S is finite and R is infinite, Eq. (3.16) takes the form

$$(V_1 - c)(w_2'' - \alpha^2 w_2) - \frac{1}{S}(V_1 - c)(w_2'''' - 2\alpha^2 w_2'' + \alpha^4 w_2) = (V_1'' - \frac{1}{S} V_1'''')w_2 . \tag{4.1}$$

We shall now prove the following theorem.

The existence of a point in the flow field for which $V_1'' - (1/S)V_1''''$ is equal to zero, is a necessary condition for the amplification of a disturbance.

Regarding w_2 as a complex variable, we define

$$M(w_2) = \frac{1}{S} w_2'''' - \left(1 + \frac{2\alpha^2}{S}\right)w_2'' + \left(\alpha^2 + \frac{\alpha^4}{S}\right)w_2 + \frac{\left(V_1'' - \frac{1}{S} V_1''''\right)}{V_1 - c} w_2 \tag{4.2}$$

and

$$\bar{M}(w_2) = \frac{1}{S} \bar{w}_2'''' - \left(1 + \frac{2\alpha^2}{S}\right)\bar{w}_2'' + \left(\alpha^2 + \frac{\alpha^4}{S}\right)\bar{w}_2 + \frac{\left(V_1'' - \frac{1}{S} V_1''''\right)}{V_1 - \bar{c}} \bar{w}_2 , \tag{4.3}$$

where a bar denotes the complex conjugate of the corresponding unbarred quantity. It is easily seen from Eq. (4.1) that both $M(w_2)$ and $\bar{M}(w_2)$ are equal to zero.

To prove the theorem we assume that $V_1'' - (1/S)V_1'''' \neq 0$ throughout the flow field. Then, multiplying $M(w_2)$ by $\bar{w}_2$ and $\bar{M}(w_2)$ by w_2 and subtracting the resulting expressions, we obtain

$$\begin{aligned} \bar{w}_2 M(w_2) - w_2 \bar{M}(w_2) = {} & \frac{1}{S}(\bar{w}_2 w_2'''' - w_2 \bar{w}_2'''') \\ & - \left(1 + \frac{2\alpha^2}{S}\right)(\bar{w}_2 w_2'' - w_2 \bar{w}_2'') \\ & + \left(V_1'' - \frac{1}{S} V_1''''\right)\left(\frac{1}{V_1 - c} - \frac{1}{V_1 - \bar{c}}\right) | w_2 |^2. \end{aligned} \tag{4.4}$$

Integrating Eq. (4.4) with respect to η_2 between the limits $\eta_2 = 0$ and $\eta_2 = 1$, we have

$$\int_0^1 [\bar{w}_2 M(w_2) - w_2 \bar{M}(w_2)]\, d\eta_2 = \frac{1}{S} [(\bar{w}_2 w_2''' - w_2 \bar{w}_2''') - (\bar{w}_2' w_2'' - w_2' \bar{w}_2'')] \Big|_0^1$$
$$- \left(1 + \frac{2\alpha^2}{S}\right)[\bar{w}_2 w_2' - w_2 \bar{w}_2'] \Big|_0^1 + 2ic_i \int_0^1 \frac{\left(V_1'' - \dfrac{1}{S} V_1''''\right)}{| V_1 - c |^2} | w_2 |^2 \, d\eta_2 . \tag{4.5}$$

Because of the boundary conditions (3.17), the first two terms on the right hand side of Eq. (4.5) vanish, and further, since both $M(w_2)$ and $\bar{M}(w_2)$ are equal to zero, the left hand side of the equation is equal to zero. It then follows that

$$c_i \int_0^1 \frac{\left(V_1'' - \dfrac{1}{S} V_1''''\right)}{| V_1 - c |^2} | w_2 |^2 \, d\eta_2$$

must vanish. But this is impossible, since $c_i > 0$, for a disturbance which tends to grow, and hence $[1/| V_1 - c |^2] > 0$. Further $| w_2 |^2$ is positive and by assumption $V_1'' - (1/S)V_1'''' \neq 0$ everywhere in the flow field. Thus we conclude that for a disturbance which tends to grow, there exists an η_2, $0 < \eta_2 < 1$, for which $V_1'' - (1/S)V_1''''$ is equal to zero.

In the case of laminar flow by a flat plate the proof of the theorem differ only in that the limits of integration become $\eta_2 = 0$ and $\eta_2 = \infty$, and the boundary conditions (3.18) are used in place of those given by (3.17).

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TRANSIENT MOTION OF A LINE LOAD ON THE SURFACE OF AN ELASTIC HALF-SPACE*

BY

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Abstract. The present paper studies the wave patterns generated in an elastic half-space by a line load moving on its surface with a velocity varying as a step function of time. The solution given in closed form is obtained by means of Fourier integral equations techniques following a Laplace transformation with respect to the time variable. The inversion of the Laplace transforms is based on a trick due to Cagniard and De Hoop.

Introduction. For the past few years, people have been interested in the effects that moving blast waves on the surface of the earth exert on its interior and in particular on underground structures. The problem has been considered by Sneddon [1] and independently by Cole and Huth [2], who treated the steady motion of a line load on the surface of an elastic half-space. Although it is clear that, far enough from the starting position of the load, the motion can be considered as essentially steady, there is one important feature that is not exhibited by a steady state solution, i.e., the "resonance" effect present in a motion at a velocity approaching the Rayleigh wave velocity. The present paper will take account of the latter effect by considering the transient motion of a line load at a velocity varying as a step function of the time.

The initial and boundary value problem. Let a Cartesian system of coordinate. x, y, z be defined such that the elastic half-space is represented by $y \geq 0$, that the axis of the line load is in the z-direction and that its initial position coincides with the z-axis. The strength of the load is assumed constant so that the problem is a two-dimensional one in x, y. Furthermore, since the medium extends indefinitely in the z-direction, the problem is one of plane strain. Then, if u and v denote the components of the displacement in the x- and y-directions respectively, it can be shown [3] that the equations of elastic motion are satisfied if

$$u = -\frac{\partial\Phi}{\partial x} + \frac{\partial\Psi}{\partial y}, \qquad v = -\frac{\partial\Phi}{\partial y} - \frac{\partial\Psi}{\partial x}, \tag{1}$$

where Φ and Ψ are solutions of

$$\nabla^2\Phi = v_1^2 \frac{\partial^2\Phi}{\partial t^2}, \qquad \nabla^2\Psi = v_2^2 \frac{\partial^2\Psi}{\partial t^2}. \tag{2}$$

In (2), v_1^2 and v_2^2 are defined as

$$v_1^2 = \rho/(\lambda + 2\mu), \qquad v_2^2 = \rho/\mu, \tag{3}$$

where ρ is the density of the material, and λ, μ its Lamé constants.

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The initial and boundary conditions are:

$$\begin{cases} \tau_{yy} = -\delta(x - c_0^{-1}t) & y = 0, x > 0 \\ \quad = 0 & y = 0, x < 0 \end{cases} \tag{4}$$

$$\tau_{xy} = 0 \qquad y = 0, \quad \text{for all} \quad x, \tag{5}$$

where τ_{yy} and τ_{xy} are respectively the normal and shear stresses at the surfaces of constant y, δ is a Dirac delta-function, and c_0^{-1} is the velocity of the moving load assumed to be a constant. We thus consider a normal load only, but it will be seen that the general case could be handled just as easily, and that the characteristic features of the wave patterns of direct interest here are not changed if we add to the load a tangential component. To the conditions (4)–(5), we must, of course, add the condition that the waves be outgoing waves. To complete the formulation of the problem, we require the wave functions Φ and Ψ to be continuously differentiable at least twice everywhere except at the the wave fronts, and the displacement to be a continuous function of the space coordinates everywhere inside the solid.

The method of solution consists in applying a Laplace transformation to the wave equations (2), and then expressing the solutions of the resulting equations as superpositions of "plane waves" with the amplitude spectra as the unknowns. Thus, under a Laplace transformation defined in the usual manner, the Eqs. (2) become:

$$\nabla^2\Phi^* = v_1^2p^2\Phi^*, \qquad \nabla^2\Psi^* = v_2^2p^2\Psi^*, \tag{6}$$

where p is the parameter of the transformation and the superscript* indicates a Laplace transform. Let the solutions of (6) be of the forms:

$$\Phi^* = \int_{-\infty}^{\infty} P(s) \exp\{-p(s^2 + v_1^2)^{1/2}y + ipsx\}\, ds, \tag{7}$$

$$\Psi^* = \int_{-\infty}^{\infty} Q(s) \exp\{-p(s^2 + v_2^2)^{1/2}y + ipsx\}\, ds, \tag{8}$$

where $P(s)$ and $Q(s)$ are unknown functions and the paths of integration are along the real axis. The functions $(s^2 + v_{1,2}^2)^{1/2}$ are given non-negative real parts along the path of integration; this can be achieved by defining their branch cuts as follows: Re $s = 0$, $v_{1,2} < |\text{Im } s| < \infty$ respectively. Finally, the parameter p in (7)–(8) is taken to be a real and positive quantity of magnitude sufficiently large to insure the convergence of the integrals.

The problem of course is to determine the functions $P(s)$ and $Q(s)$ from the boundary conditions. Now, the conditions of (4) and (5) can be expressed in terms of Φ and Ψ using (1) and the well-known strain-displacement and stress-strain relations. Assuming that (7) and (8) can be differentiated under the integral signs, we accordingly obtain for the Laplace transforms of (4)–(5)

$$\int_{-\infty}^{\infty} \{(s^2 + v_2^2/2)P(s) - is(s^2 + v_1^2)^{1/2}Q(s)\} \exp(ipsx)\, ds$$
$$\left.\begin{aligned} &= (c_0v_2^2/2\rho p^2)\exp(-pc_0x) \quad \text{for} \quad x > 0 \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad x < 0 \end{aligned}\right\}, \tag{9}$$

$$\int_{-\infty}^{\infty} \{2is(s^2 + v_1^2)^{1/2}P(s) + (2s^2 + v_2^2)Q(s)\} \exp(ipsx)\, ds = 0 \tag{10}$$

the latter equation holding for all x. Equation (10) is evidently satisfied if the integrand is identically zero, i.e., if

$$P(s) = (s^2 + v_2^2/2)R(s), \qquad Q(s) = -is(s^2 + v_1^2)^{1/2}R(s), \tag{11}$$

where $R(s)$ is still largely arbitrary. Substituting for $P(s)$ and $Q(s)$ into (9) gives:

$$\left.\begin{aligned}\int_{-\infty}^{\infty} F(s)R(s) \exp(ipsx)\, ds &= (A_0/p^2) \exp(-pc_0x), \qquad x > 0\\ &= 0 \qquad\qquad\qquad\qquad\qquad\; x < 0\end{aligned}\right\}, \tag{12}$$

where

$$A_0 \equiv c_0v_2^2/2\rho, \qquad F(s) \equiv (s^2 + v_2^2/2)^2 - s^2(s^2 + v_1^2)^{1/2}(s^2 + v_2^2)^{1/2}. \tag{13}$$

The solution of (12) is immediate

$$R(s) = (A_0/2\pi ip^2)\{(s - ic_0)F(s)\}^{-1}. \tag{14}$$

From (11) and (14), the expressions for $P(s)$ and $Q(s)$ can be deduced and the problem is formally solved. It is not difficult to show that this solution converges and is the actual solution of the boundary value problem.

The stress wave patterns. It remains to find the inverse transform of the solutions. The Laplace transforms of the stresses are given in terms of integrals of the form

$$I^* = \int_{-\infty}^{\infty} K(s) \exp\{-p(s^2 + v^2)^{1/2}y + ipsx\}\, ds, \tag{15}$$

where $v = v_{1,2}$ and $K(s)$ is some algebraic function of s only. The present inversion of (15) back to the t-plane is based on a trick originally due to Cagniard [4] and modified by De Hoop [5]. The trick consists in so deforming the path of integration that I^* is reduced to the form

$$I^* = \int_0^{\infty} J(t; x, y)e^{-pt}\, dt, \tag{16}$$

where t is real and $J(t; x, y)$ is some suitable function. The inverse of I^* can thus be obtained by inspection. In the course of the deformation of the path of integration, it is essential to know all the singularities of $K(s)$. These can be recognized by inspection, with the exception of the poles $s = \pm is_R$, $s_R > v_2$, $\pm is_R$ being the zeros* of $F(s)$ defined in (13). Once the singularities of $K(s)$ are known, it only remains to follow steps quite similar to those taken in [6] where the inverse transforms of integrals of the form (15) have been found. We refer the reader to [6] for these details and are content to state here the final expressions for the stresses corresponding to a velocity of the load smaller** than both velocities of sound in the solid

*Thus s_R^{-1} is the well-known Rayleigh wave velocity.

**The case of a load velocity greater than either or both velocities of sound could be handled just as easily but is omitted for simplicity.

$$\tau_{yy} = -\frac{c_0}{\pi r} H(t - v_1 r)\left\{\frac{t \sin\theta}{(t^2 - v_1^2 r^2)^{1/2}} \operatorname{Re} \frac{M_1(s_1)}{(c_0 + is_1)(s_R + is_1)} - \operatorname{Im} \frac{M_1(s_1)\cos\theta}{(c_0 + is_1)(s_R + is_1)}\right\} + \Delta\tau, \tag{17}$$

$$\Delta\tau \equiv \frac{c_0}{\pi r} H(t - v_2 r)\left\{\frac{t \sin\theta}{(t^2 - v_2^2 r^2)^{1/2}} \operatorname{Re} \frac{M_2(s_2)}{(c_0 + is_2)(s_R + is_2)} - \operatorname{Im} \frac{M_2(s_2)\cos\theta}{(c_0 + is_2)(s_R + is_2)}\right\}$$
$$- \frac{c_0}{\pi r} f(t) g(\theta)\left\{\pm\frac{t \sin\theta}{(v_2^2 r^2 - t^2)^{1/2}} + \cos\theta\right\}\cdot \operatorname{Im} \frac{M_2(\omega_*)}{(c_0 + i\omega_*)(s_R + i\omega_*)},$$

$$\tau_{xx} = \frac{c_0}{\pi r} H(t - v_1 r)\left\{\frac{t \sin\theta}{(t^2 - v_1^2 r^2)^{1/2}} \operatorname{Re} \frac{N_1(s_1)}{(c_0 + is_1)(s_R + is_1)} - \operatorname{Im} \frac{N_1(s_1)\cos\theta}{(c_0 + is_1)(s_R + is_1)}\right\} - \Delta\tau, \tag{18}$$

$$\begin{aligned}\tau_{xy} = -\frac{c_0}{\pi r} H(t - v_1 r)&\left\{\frac{t \sin\theta}{(t^2 - v_1^2 r^2)^{1/2}} \operatorname{Re} \frac{N_2(s_1)}{(c_0 + is_1)(s_R + is_1)} - \operatorname{Im} \frac{N_2(s_1)\cos\theta}{(c_0 + is_1)(s_R + is_1)}\right\} \\ &+ \frac{c_0}{\pi r} H(t - v_2 r)\left\{\frac{t \sin\theta}{(t^2 - v_2^2 r^2)^{1/2}} \operatorname{Re} \frac{N_2(s_2)}{(c_0 + is_2)(s_R + is_2)} - \operatorname{Im} \frac{N_2(s_2)\cos\theta}{(c_0 + is_2)(s_R + is_2)}\right\} \\ &- \frac{c_0}{\pi r} f(t) g(\theta)\left\{\pm\frac{t \sin\theta}{(v_2^2 r^2 - t^2)^{1/2}} + \cos\theta\right\} \operatorname{Im} \frac{N_2(\omega_*)}{(c_0 + i\omega_*)(s_R + i\omega_*)},\end{aligned} \tag{19}$$

where

$$r = (x^2 + y^2)^{1/2}, \qquad \theta = \tan^{-1}(y/x), \qquad 0 \le \theta \le \pi, \tag{20}$$

$$s_{1,2} = (t^2/r^2 - v_{1,2}{}^2)^{1/2} \sin\theta + i(t/r)\cos\theta, \tag{21}$$

$$M_1(s) = (s^2 + v_2^2/2)^2 (s_R + is)/F(s), \tag{22}$$

$F(s)$ being defined in (13)

$$M_2(s) = s^2 (s^2 + v_1^2)^{1/2} (s^2 + v_2^2)^{1/2} (s_R + is)/F(s), \tag{23}$$

$$(s_{1,2}^2 + v_{1,2}^2)^{1/2} = \{t \sin\theta + i(t^2 - v_{1,2}^2 r^2)^{1/2} \cos\theta\}/r, \tag{24}$$

$$(s_{1,2}^2 + v_{2,1}^2)^{1/2} = \left\{\frac{(A_{1,2}^2 + B_{1,2}^2)^{1/2} + A_{1,2}}{2}\right\}^{1/2} \pm i\left\{\frac{(A_{1,2}^2 + B_{1,2}^2)^{1/2} - A_{1,2}}{2}\right\}^{1/2}, \tag{25}$$

$$A_{1,2} \equiv (t^2/r^2)(\sin^2\theta - \cos^2\theta) + v_{2,1}^2 - v_{1,2}^2 \sin^2\theta,$$

$$B_{1,2} \equiv (t^2/r^2 - v_{1,2}^2)^{1/2} (t/r) \sin 2\theta,$$

the $\pm$ signs in (25) corresponding to $\cos\theta > 0$ and $\cos\theta < 0$ respectively.

$$\begin{cases} f(t) = 1 \quad t_S \le t \le v_2 r \\ \qquad t_S \equiv v_2 r \mid \cos\theta \mid + (v_2^2 - v_1^2)^{1/2} r \sin\theta \\ \quad = 0 \text{ otherwise} \end{cases} \tag{26}$$

$$\begin{cases} g(\theta) = 1 \quad 0 \le \theta \le \theta_S \quad \text{or} \quad \pi - \theta_S \le \theta \le \pi \\ \qquad \theta_S \equiv \cos^{-1}(v_1/v_2) \\ \quad = 0 \text{ otherwise} \end{cases} \tag{27}$$

$$\omega_\pm = i\{\mp(v_2^2 - t^2/r^2)^{1/2} \sin\theta + (t/r)\cos\theta\}, \tag{28}$$

$\omega_\pm$ corresponding to $0 \le \theta \le \theta_S$ and $\pi - \theta_S \le \theta \le \pi$ respectively.

$$(\omega_\pm^2 + v_1^2)^{1/2} = \pm i(\mid \omega_\pm \mid^2 - v_1^2)^{1/2}, \tag{29}$$

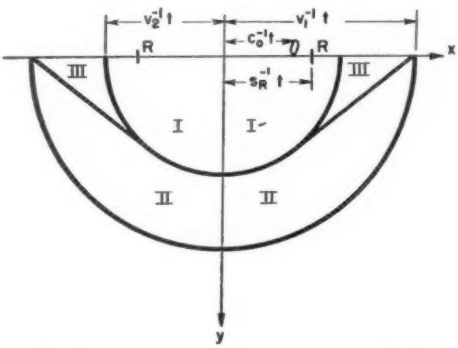
$$(\omega_\pm^2 + v_2^2)^{1/2} = (v_2^2 - \mid \omega_\pm \mid^2)^{1/2}, \tag{30}$$

$$N_1(s) = (s^2 + v_1^2 - v_2^2/2)(s^2 + v_2^2/2)(s_R + is)/F(s), \tag{31}$$

$$N_2(s) = is(s^2 + v_1^2)^{1/2}(s^2 + v_2^2/2)(s_R + is)/F(s). \tag{32}$$

In the above relations the square roots of real quantities are taken as positive.

From (17)–(19), it is clear that the wave patterns consist of two cylindrical waves radiating from the origin $r = 0$ with velocities v_1^{-1} and v_2^{-1} respectively, two head waves whose fronts are tangential to the shear cylindrical wave—the head waves are represented e.g. in $\Delta\tau$ in (18)—the two surface waves propagating in the positive and negative x-directions respectively. The complete picture is shown in the figure below.



Stress wave patterns. I. Shear wave, II. Dilatation wave, III. Head waves, R. Rayleigh wave.

It can be verified that the stresses are of the order $(vr^2 - t^2)^{-1/2}$, for $vr \to t$, $v = v_{1,2}$. This singularity is, of course, integrable and does not affect the continuity of the displacements. The resonance effect anticipated in the Introduction results from the

presence of the factors $(c_0 + is)^{-1}$ $(s_R + is)^{-1}$, $s = s_{1,2}$, in the expressions (17)-(19) since $- is_R$ is not a zero of the functions $M(s)$, $N(s)$ defined above.

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THE STOKES FLOW ABOUT A SPINDLE*

BY

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I. Statement of the problem for an axially symmetric body. The Stokes flow of a viscous, incompressible fluid about a body is defined by the assumption that inertial effects are negligible in comparison with those of viscosity, or, more precisely, that the Reynolds number of the flow is small. In the case in which the flow is two-dimensional or has radial symmetry, the introduction of a stream function and the Stokes assumption together with the no-slip boundary condition on the body and an appropriate assumption at infinity, yields a boundary value problem which has been solved in a number of instances. See Dryden, Murnaghan and Bateman [1], pp. 295-312 and the references of [2].

In the axi-symmetric case a body (or configuration of bodies) having an axis of symmetry is immersed in a flow which has a uniform velocity parallel to the axis of symmetry. It is then reasonable to suppose that the resulting flow pattern is identical in all planes through the axis of symmetry. If we introduce cylindrical coordinates (x, r, θ) into the flow space, where $x(-\infty \leq x \leq \infty)$ is measured along the axis of symmetry, $r(0 \leq r \leq \infty)$ normal to this axis, and $\theta(0 \leq \theta < 2\pi)$ is measured with respect to an arbitrary plane through the axis of symmetry, then θ plays no further role in our analysis because of the axial symmetry. Now let the velocity of the fluid be $\mathbf{u}(x, r) = (u_x(x, r), u_r(x, r))$ and introduce a stream function $\psi(x, r)$ by the equations

$$u_x = \frac{1}{r}\frac{\partial \psi}{\partial r}, \qquad u_r = -\frac{1}{r}\frac{\partial \psi}{\partial x}. \tag{1.1}$$

By a well-known procedure (Payne and Pell [2]; Milne-Thomson [3], pp. 521-523) the differential equation to be satisfied in the region of flow D is found to be

$$L_{-1}^2 \psi = 0, \tag{1.2}$$

where

$$L_{-1} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}. \tag{1.3}$$

Let the trace of the boundary of the body in a meridional plane be C (Fig. 1). Then the condition of vanishing velocity on C can be stated in the form

$$\left.\begin{aligned} \psi &= 0 \qquad (1.4) \\ \frac{\partial \psi}{\partial n} &= 0 \qquad (1.5) \end{aligned}\right\} \quad \text{on} \quad C,$$

where $\mathbf{n}$ is the unit normal to C exterior to the body.

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If the uniform velocity of the flow at infinity is $\mathbf{u} = (U, 0)$, then ψ must satisfy the condition

$$\lim_{\rho\to\infty} \psi = \tfrac{1}{2}r^2U + O(\rho), \tag{1.6}$$

where

$$\rho^2 = x^2 + r^2. \tag{1.7}$$

II. Representation of the solution. It is expedient to define a second stream function ψ_1 by the relation

$$\psi = \tfrac{1}{2}Ur^2 - \psi_1\,. \tag{2.1}$$

We then find that ψ_1 must satisfy the equation

$$L_{-1}^2\psi_1 = 0 \tag{2.2}$$

in D, subject to the conditions

$$\psi_1 = \tfrac{1}{2}Ur^2, \tag{2.3}$$

$$\frac{\partial\psi_1}{\partial n} = Ur\,\frac{\partial r}{\partial n}, \tag{2.4}$$

on C, as well as the condition that ψ_1 give rise to a vanishing velocity at infinity.

Following Weinstein [4], solutions of

$$L_k(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{k}{r}\frac{\partial v}{\partial r} = 0 \tag{2.5}$$

k real, are known as generalized axially symmetric potentials, and denoted by ψ^k. Payne [5] has shown that in certain regions any solutions of the repeated operator equation (2.2) can be represented as a linear combination of any two of the functions

a) $r^2\psi^3$, b) $xr^2\psi^3$, c) $\rho^2r^2\psi^3$,

d) $r^2\psi^1$, e) $r^4\psi^5$.

In a previous paper [2] the authors have used this theorem to obtain the solution of certain problems in Stokes flow; it will be used here to obtain still another.

III. The flow about a spindle. We introduce bipolar coordinates (ξ, η) into the (x, r) plane by the transformation

$$z = ib \cot(\zeta/2), \tag{3.1}$$

where $z = x + ir$, $\zeta = \xi + i\eta$, and $b > 0$ is a constant. In terms of the (ξ, η) coordinates

$$x = \frac{b \sinh \eta}{\cosh \eta - \cos \xi}, \qquad r = \frac{b \sin \xi}{\cosh \eta - \cos \xi}, \tag{3.2}$$

where $-\infty < \eta < \infty$ and $0 \le \xi \le \pi$. We define a spindle to be an object whose surface is obtained by revolving the curve $\xi = \xi_0 (0 < \xi_0 < \pi)$ about the x axis. This curve is the arc lying in $r \ge 0$ of the circle which passes through $(\pm b, 0)$ and has its center at $(0, -b \cot \xi_0)$. The surface of the spindle is thus $\xi = \xi_0$ and the region of flow D is $0 \le \xi \le \xi_0$, $-\infty < \eta < \infty$ (Fig. 1). It is advantageous in considering flow about the spindle to choose ξ and η as independent variables.

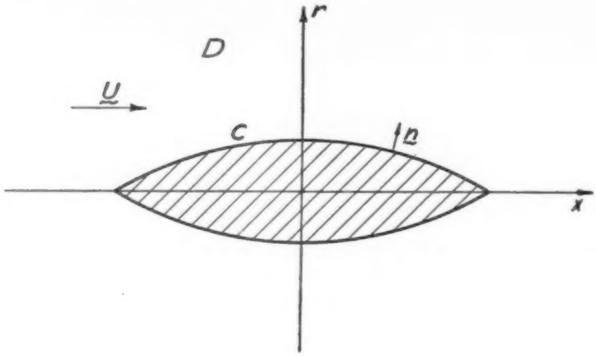

FIG. 1.

Using the results of [5] we represent the solution of (2.2) in the form

$$\psi_1 = r^2\psi^1 + (\rho^2 - b^2)r^2\psi^3. \tag{3.3}$$

It is not difficult to see [6] that any $\psi^{2n+1}(\xi, \eta)$ which is even in η may be expressed in the form

$$\psi^{2n+1} = (\cosh \eta - \cos \xi)^{(2n+1)/2} \int_0^\infty F(\alpha)K_\alpha^{(n)} (\cos \xi) \cos \alpha\eta \, d\alpha, \tag{3.4}$$

where $F(\alpha)$ is an arbitrary function of α,

$$K_\alpha (\cos \xi) = P_{i\alpha - 1/2} (\cos \xi) \tag{3.5}$$

(known as the conal function; see Hobson [7], pp. 444-453) and $K_\alpha^{(n)}(\lambda) \equiv d^n K_\alpha(\lambda)/d\lambda^n$. Substitution of (3.4) in (3.3) permits us to write ψ_1 in the form

$$\psi_1 = \tfrac{1}{2}Ur^2(s - t)^{1/2} \int_0^\infty [A(\alpha)tK_\alpha^{(1)}(t) + B(\alpha)K_\alpha(t)] \cos \alpha\eta \, d\alpha, \tag{3.6}$$

where

$$s = \cosh \eta \qquad t = \cos \xi \tag{3.7}$$

and $A(\alpha)$, $B(\alpha)$ are functions to be determined in such a way that the boundary conditions (2.3-.4) are satisfied. The first of these yields

$$\int_0^\infty [A(\alpha)t_0K_\alpha^{(1)}(t_0) + B(\alpha)K_\alpha(t_0)] \cos \alpha\eta \, d\alpha = (s - t_0)^{-1/2}. \tag{3.8}$$

Next we observe that $\partial(\ )/\partial n = 0$ is equivalent to $\partial(\ )/\partial\xi = 0$ on $\xi = \xi_0$, and make use of (3.8) to show that (2.4) can be written in the form

$$\int_0^\infty \left\{A(\alpha) \frac{\partial}{\partial t_0} [t_0K_\alpha^{(1)}(t_0)] + B(\alpha) \frac{\partial K_\alpha(t_0)}{\partial t_0}\right\} \cos \alpha\eta \, d\alpha = \frac{\partial}{\partial t_0} (s - t_0)^{-1/2}. \tag{3.9}$$

The conal function $K_\alpha(-t)$ may be defined ([7], p. 446) by

$$K_\alpha(-t) = \frac{2^{1/2}}{\pi} \cosh \alpha\pi \int_0^\infty [\cosh u - \cos \xi]^{-1/2} \cos \alpha u \, du. \tag{3.10}$$

But for $0 < \xi_0 < \pi$, $(\cosh \eta - \cos \xi_0)^{-1/2}$ satisfies the hypotheses of the Fourier integral theorem, and utilizing (3.10) we thus have

$$(s - t_0)^{-1/2} = \frac{2}{\pi} \int_0^\infty \cos \alpha\eta \left\{ \int_0^\infty [\cosh u - t_0]^{-1/2} \cos \alpha u \, du \right\} d\alpha \\ = 2^{1/2} \int_0^\infty \frac{K_\alpha(-t_0) \cos \alpha\eta}{\cosh \alpha\pi} \, d\alpha, \tag{3.11}$$

where $t_0 = \cos \xi_0$. Thus the right hand member of (3.8) may be replaced by the integral (3.11). For $0 < \xi_0 < \pi$ one may differentiate (3.11) with respect to t_0, and obtain a representation for the right-hand member of (3.9). We are led in this way to two linear equations for A and B whose solution is

$$A(\alpha) = \frac{2^{1/2}}{\Omega \cosh \alpha\pi} [K_\alpha(-t_0) K_\alpha^{(1)}(t_0) - K_\alpha^{(1)}(-t_0) K_\alpha(t_0)], \tag{3.12}$$

$$B(\alpha) = \frac{2^{1/2}}{\Omega \cosh \alpha\pi} \left[t_0 K_\alpha^{(1)}(t_0) K_\alpha^{(1)}(-t_0) - K_\alpha(-t_0) \frac{d}{dt_0} (t_0 K_\alpha^{(1)}(t_0))\right], \tag{3.13}$$

where

$$\Omega = t_0 [K_\alpha^{(1)}(t_0)]^2 - K_\alpha(t_0) \frac{d}{dt_0} (t_0 K_\alpha^{(1)}(t_0)). \tag{3.14}$$

More compact expressions for A and B can be obtained, however. Suitable regrouping of the terms of Ω and repeated use of the fact that K_α satisfies a Legendre equation yields

$$\frac{d}{dt} [(1 - t^2)\Omega] = -2K_\alpha(t) K_\alpha^{(2)}(t). \tag{3.15}$$

Integration of this with respect to t and division of the result by $1 - t^2$ gives

$$\Omega = \frac{2}{1 - t_0^2} \int_{t_0}^1 K_\alpha(\tau) K_\alpha^{(2)}(\tau) \, d\tau, \qquad -1 < t_0 < 1. \tag{3.16}$$

The bracketed quantity in A is simply the Wronskian of $K_\alpha(-t)$ and $K_\alpha(t)$ evaluated at t_0, which (Neumann [8], pp. 207-210) is $-2 \cosh \alpha\pi/\pi(1 - t_0^2)$. Thus we obtain from (3.12)

$$A(\alpha) = -\frac{2^{1/2}}{\pi} \left(\int_{t_0}^1 K_\alpha(\tau) K_\alpha^{(2)}(\tau) \, d\tau \right)^{-1} \tag{3.17}$$

for $-1 < t_0 < 1$.

By essentially the same procedure as was used in discussing Ω, it can be shown that

$$\frac{d}{dt} \left\{ (1 - t^2)\left[t K_\alpha^{(1)}(t) K_\alpha^{(1)}(-t) - K_\alpha(-t) \frac{d}{dt} (t K_\alpha^{(1)}(t))\right] \right\} = -2K_\alpha^{(2)}(t) K_\alpha(-t) \tag{3.18}$$

holds for $-1 < t < 1$. This may be integrated with respect to t from t_0 $(-1 < t_0 \leq 1)$ to 1 and divided by $1 - t_0^2$ in order to obtain the bracketed portion of $B(\alpha)$. It will be noted that (3.18) is not valid at $t = \pm 1$, since these are singular points for either $K_\alpha(t)$ or $K_\alpha(-t)$ and their derivatives, but the relations ([8], pp. 207 and 209)

$$\lim_{t\to 1} (1-t)^j K_\alpha^{(j)}(-t) = \frac{\Gamma(j)}{\pi} \cosh \alpha\pi, \qquad j = 1, 2, \cdots \tag{3.19}$$

$$K_\alpha(1) = 1 \tag{3.20}$$

$$K_\alpha^{(j)}(1) = \frac{1}{2^j \Gamma(j+1)} \left[\alpha^2 + \left(\frac{1}{2}\right)^2\right]\left[\alpha^2 + \left(\frac{3}{2}\right)^2\right] \cdots \left[\alpha^2 + \left(\frac{j-1}{2}\right)^2\right],$$
$$j = 1, 2, \cdots \tag{3.21}$$

permit the integration just mentioned to be carried out up to $t = 1$. We find then that (3.13) yields

$$B(\alpha) = A(\alpha)\left\{\frac{1}{2}\left(\alpha^2 + \frac{1}{4}\right) - \frac{\pi}{\cosh \alpha\pi} \int_{t_0}^{1} K(-\tau) K^{(2)}(\tau)\, d\tau\right\} \tag{3.22}$$

for $-1 < t_0 < 1$.

The insertion of (3.17) and (3.22) in (3.6), and of the result in (2.1) gives the stream function for the Stokes flow about the spindle.

It is easily verified that for $\xi_0 = \pi/2 (t_0 = 0)$, Eq. (3.6) with $A(\alpha)$ and $B(\alpha)$ defined by (3.12) and (3.13) [or by (3.17) and (3.22)] gives the correct result for the flow about a sphere. In this case we note that

$$\lim_{t_0\to 0} K_\alpha(t_0) = \lim_{t_0\to 0} K_\alpha(-t_0) \tag{3.23}$$

and

$$\lim_{t_0\to 0} K^{(1)}(t_0) = -\lim_{t_0\to 0} K^{(1)}(-t_0). \tag{3.24}$$

For $t_0 = 0$, (3.12) and (3.13) then gives

$$A(\alpha) = -2^{3/2}/\cosh \alpha\pi, \tag{3.25}$$

$$B(\alpha) = 2^{1/2}/\cosh \alpha\pi. \tag{3.26}$$

From (3.6), the expression for ψ_1 becomes

$$\psi_1 = 2^{1/2} U r^2 (s-t)^{1/2} \int_0^\infty \frac{[\frac{1}{2}K_\alpha(t) - tK_\alpha^{(1)}(t)]}{\cosh \alpha\pi} \cos \alpha\eta\, d\alpha. \tag{3.27}$$

The integral on the right hand side of (3.27) is easily evaluated from (3.11) and the expression obtained by differentiation of (3.11) with respect to t_0. We merely replace ξ_0 by $\pi - \xi$ in the resulting integral formulas and obtain at once the following expression for ψ_1:

$$\psi_1 = \tfrac{1}{2} U r^2 (s-t)^{1/2} \{(s+t)^{-1/2} + t(s+t)^{-3/2}\}. \tag{3.28}$$

This may be rewritten as

$$= \tfrac{1}{4} U r^2 (s-t)^{1/2} \{3(s+t)^{-1/2} - (s-t)(s+t)^{-3/2}\}$$

$$= \tfrac{1}{4} U r^2 \{3b\rho^{-1} - b^3 \rho^{-3}\} \tag{3.29}$$

where b is the radius of the sphere. This is the well known result of Stokes (see [3]).

IV. Drag of the spindle. It was shown in [2] that the drag P of an axially symmetric body for which the flow region D is simply connected is given by

$$\frac{P}{8\pi\mu} = \lim_{\rho\to\infty} \frac{\rho\psi_1}{r^2}. \tag{4.1}$$

Substituting from (1.7), (3.2), and (3.6) in (4.1) we obtain

$$\frac{P}{8\pi\mu} = \frac{Ub}{2} \lim_{\substack{\xi\to 0\\ \eta\to 0}} (s+t)^{1/2} \int_0^\infty [A(\alpha)tK_\alpha^{(1)}(t) + B(\alpha)K_\alpha(t)] \cos\alpha\eta \, d\alpha,$$

$$= \frac{2^{1/2}}{2} Ub \int_0^\infty [A(\alpha)K_\alpha^{(1)}(1) + B(\alpha)K_\alpha(1)] \, d\alpha.$$

From (3.20-.21) we obtain $K_\alpha(1)$ and $K_\alpha^{(1)}(1)$, and thus the drag of the spindle becomes

$$P = 2^{3/2}\pi\mu bU \int_0^\infty [2B(\alpha) - (\alpha^2 + \tfrac{1}{4})A(\alpha)] \, d\alpha \tag{4.2}$$

or

$$P = 8\pi\mu bU \int_0^\infty \frac{F(\alpha)}{\cosh\alpha\pi} \, d\alpha, \tag{4.3}$$

where

$$F(\alpha) = \int_{t_0}^1 K_\alpha(-\tau)K_\alpha^{(2)}(\tau) \, d\tau \Big/ \int_{t_0}^1 K_\alpha(\tau)K^{(2)}(\tau) \, d\tau. \tag{4.4}$$

With $A(\alpha)$ and $B(\alpha)$ defined by (3.25) and (3.26), Eq. (4.2) gives the well known result for the sphere.

The reviewer has kindly called our attention to the fact that tables of the functions $K_\alpha(t)$ are now being compiled (M. I. Zhurina and L. N. Karmazina, *Tablitsky funktsii Lezhandra* $P_{-1/2+i\tau}(x)$.) These tables should facilitate the computation of the drag coefficient P as a function of ξ_0.

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ON PARAMOUNT MATRICES*

BY

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Introduction. A paramount matrix is a symmetric matrix of real numbers such that any principal minor must be at least as large as the absolute value of any other minor of the same order built from the same rows. In this paper a new property of paramount matrices is cited and proved, and some remarks are made relevant to the problem of the synthesis of multiport resistive networks. Unfortunately, to achieve a concise and rigorous presentation, a rather formal mathematical notation was developed, and some of the results, although not particularly profound, may be obscured by the mathematical formalism. Hence, a long introduction is included which describes the results in less exact but more comprehensible terms.

After several paragraphs introducing the notation, it is shown in Theorem 9 that if any elements in the main diagonal of a paramount matrix are increased and all other elements remain fixed, the resulting new matrix is still paramount. The remainder of the paper is devoted to the problem of the synthesis of multiport resistive networks.

It is well known [1] that if A is either the open-circuit impedance matrix or short circuit admittance matrix of a multiport resistive network, then A is paramount. However, it is not known whether paramountcy is sufficient for the realization of such a network. More precisely, if A is an arbitrary** paramount matrix of order m, it is not known whether there exists an m-port resistive network such that A is either the open-circuit impedance matrix or short-circuit admittance matrix of this network. Tellegen and Elias [2] have shown that the preceding statement is true if $m \leq 3$, but when m is arbitrary, the answer to the question remains one of the leading unsolved problems. This paper does not attempt to answer the question for an arbitrary m, but it does indicate a technique suggested by the the preceding result which may ultimately help to solve the problem.

Observe that an arbitrary mth order symmetric paramount matrix A has $m(m + 1)/2$ independent elements. Thus, an m-port resistive network whose open-circuit impedance matrix or short-circuit admittance matrix is A can be expected to contain, in general, at least $m(m + 1)/2$ resistive branches.

Note, however, that by Theorem 9, the set of all real numbers x such that a paramount matrix results when the element A_{11} of A is replaced by x, is a semi-infinite closed interval bounded on the left. Let b be the least element of this interval and let B be the matrix obtained from A when A_{11} is replaced by b. Now, if there exists a resistive network whose open-circuit impedance matrix is B, it is reasonable to expect that there exists such a network with $(m(m + 1)/2) - 1$ resistive branches since the number of independent elements of B is $(m(m + 1)/2) - 1$. Furthermore, if such a network is available and a resistance of $A_{11} - b$ ohms is introduced in series with port 1, then a resistive

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**An arbitrary paramount matrix of order m is a paramount matrix with exactly $m(m + 1)/2$ independent elements.

network is obtained whose open-circuit impedance matrix is A. Thus, to realize A, we need consider only geometrical structures realizing B, with $(m(m+1)/2) - 1$ resistive branches. This idea can be generalized as follows.

In this paper, in Sec. 17, a number $n(A)$ is defined for any paramount matrix A. This number $n(A)$ is the maximum number such that the realization of A can be achieved by combining $n(A)$ resistances in series or parallel with the ports of a resistive network of

$$\frac{[\text{order }(A)][1 + \text{order }(A)]}{2} - n(A)$$

resistive branches that realizes a certain minimal paramount matrix obtained from A. This minimal paramount matrix obtained from A is minimal in the sense that it is irreducible as defined in Sec. 14. Thus, realization of A has been simplified to the consideration of the realization of such a minimal paramount matrix by a resistive structure with only

$$\frac{[\text{order }(A)][1 + \text{order }(A)]}{2} - n(A)$$

resistive branches. This represents a simpler geometrical realization problem, and because of the maximality of $n(A)$, it is the best simplification which can be effected by utilizing the concept described above.

It is easy to show that for any arbitrary third-order paramount matrix A, $n(A) = 3$. Thus as shown by Elias and Telegen [2], any arbitrary third-order paramount matrix A is always realizable as the open-circuit impedance matrix of a resistive network of six branches which has three resistive branches in series or parallel with the ports and three more resistive branches realizing the minimal* paramount matrix obtained from A. The same geometrical structure will always produce a resistive three-port network whose open-circuit impedance matrix is equal to an arbitrary third-order paramount matrix A.

Unfortunately, the situation is more complicated in the case of an arbitrary fourth-order paramount matrix. It is not true that for each arbitrary fourth-order paramount matrix A, the number $n(A)$ is the same. It is easy to exhibit such a matrix A with $n(A) = 2$ and another such matrix A with $n(A) = 4$. Clearly the minimal paramount matrix resulting in the former case will have eight independent elements, while the minimal paramount matrix resulting in the latter case will have six independent elements. This leads the writer to conjecture that if paramountcy is indeed a sufficient condition for the realization of a fourth-order matrix as the open-circuit impedance matrix of a resistive network, the same geometrical structure will not suffice for each arbitrary fourth-order paramount matrix A but will depend upon the number $n(A)$. The formal mathematics follows.

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*Consider the subnetwork of Tellegen's network which does not contain any of the three resistances in series or in parallel with the ports; agree to call any three-branch resistive three-port network a minimal resistive three-port, if and only if it has the geometrical configuration of this subnetwork. It was observed by Prof. R. M. Foster of the Polytechnic Institute of Brooklyn that a third-order paramount matrix is realizable as the open-circuit impedance matrix of a minimal resistive three-port network, if and only if the matrix is irreducible (as defined in Sec. 14).

1. Definitions.

(i) $R = \{x \mid x \text{ is a real number}\}$
(ii) $\omega = \{x \mid x \text{ is a positive integer}\}$
(iii) $\operatorname{dmn} f = \{x \mid \text{for some } y, (x, y) \in f\}$
(iv) $\operatorname{rng} f = \{x \mid \text{for some } y, (y, x) \in f\}$
(v) $f \mid A = f \cap \{(x, y) \mid x \in A\}$
(vi) If $x \in \omega$, then

$$\langle x \rangle = \omega \cap \{y \mid y \leq x\}$$

(vii) pA = the cardinal number of A.

2. Definitions.

(i) For each $x \in \omega$
$S^x = \{u \mid u \text{ is an increasing sequence with } 1 < pu \text{ and } \operatorname{rng} u \subset \langle x \rangle\}$,
(ii) For each $x \in \omega$ and $i \in \langle x \rangle$

$$S_i^x = S^x \cap \{u \mid i \in \operatorname{rng} u\}.$$

3. Definitions.

(i) $M = \{A \mid A \text{ is a function, } \operatorname{dmn} A = (\langle m \rangle \times \langle m \rangle) \text{ for some } m \in \omega \text{ and } \operatorname{rng} A \subset R\}$.
(ii) If $A \in M$ and $(i, j) \in \operatorname{dmn} A$,

$$A_{ij} = A(i, j).$$

(iii) If $A \in M$, order $(A) = (p \operatorname{dmn} A)^{1/2}$.
(iv) $K = M \cap \{A \mid A_{ji} = A_{ij} \text{ for } (i, j) \in \operatorname{dmn} A \text{ and } 2 \leq \text{order } (A)\}$.
(v) For $m \in \omega$,

$$M^m = M \cap \{A \mid m = \text{order } (A)\}.$$

(vi) If $A \in M$, det (A) is the customary determinant of the square matrix A.
(vii) If $A \in M$ and det $(A) \neq 0$, then A^{-1} is the customary inverse of A.

4. Definitions.

If $u \in S^x$,

$$[u * x]$$

is that increasing sequence of positive integers such that

$$\begin{aligned} \operatorname{dmn}[u * x] &= \langle x - pu \rangle \\ [u * x]_1 &= \inf\,(\langle x \rangle - \operatorname{rng} u) \\ [u * x]_i &= \inf\,(\langle x \rangle - \operatorname{rng} u - \operatorname{rng}([u * x] | \langle i - 1 \rangle)) \\ &\quad \text{for } i \in \langle x - pu \rangle \quad \text{and} \quad i \neq 1. \end{aligned}$$

5. Definitions.

(i) For $A \in M$

$$G(A) = (S^{\text{order}(A)} \times S^{\text{order}(A)}) \cap \{(u, v) \mid pu = pv\}.$$

(ii) For $A \in M$ and $i \in \langle \text{order } (A) \rangle$

$$G_i(A) = G(A) \cap (S_i^{\text{order}(A)} \times S^{\text{order}(A)}).$$

6. Definitions.

For $A \varepsilon M$ and $(u, v) \varepsilon G(A)$

$$A^u_v \text{ is that element of } M^{\text{order}(A) - pu}$$

such that

$$(A^u_v)_{ij} = A([u * \text{order}\,(A)]_i\,,\ [v * \text{order}\,(A)]_j)$$

for

$$(i, j) \varepsilon (\langle \text{order}\,(A) - pu \rangle \times \langle \text{order}\,(A) - pu \rangle).$$

7. Definition.

A is paramount if and only if $A \varepsilon K$ and

$$\det\,(A^u_u) - |\det\,(A^u_v)| \geq 0$$

whenever $(u, v) \varepsilon G(A)$.

8. Definition.

If $A \varepsilon M$, $x \varepsilon R$ and $i \varepsilon \langle \text{order}\,(A) \rangle$, then

$$[A, i, x]$$

is that element of $M^{\text{order}(A)}$ such that

$$[A, i, x]_{jk} = A_{jk}$$

if $(j, k) \varepsilon$ dmn A and $(j, k) \neq (i, i)$.

$$[A, i, x]_{ii} = x.$$

9. Theorem.

Suppose that A is paramount and that $i \varepsilon \langle \text{order}\,(A) \rangle$. Let

$$B = \{x \mid [A, i, x] \text{ is paramount}\}.$$

Then B is a closed, semi-infinite interval, bounded on the left.

Proof.

Let $0 < t \varepsilon R$. It suffices to show that $[A, i, A_{ii} + t]$ is paramount. Pick any $(u, v) \varepsilon G_i([A, i, A_{ii} + t])$.

It suffices to show that

$$\det\,([A, i, A_{ii} + t]^u_u) \geq |\det\,([A, i, A_{ii} + t]^u_v)|.$$

Suppose first that

$$pu = pv = \text{order}\,(A) - 1.$$

In this case

$$\det\,([A, i, A_{ii} + t]^u_u) = A_{ii} + t.$$

Also, if $i \not\varepsilon$ rng v, then

$$\det([A, i, A_{ii} + t]^u_v) = A_{ii} + t;$$

while if $i \varepsilon$ rng v, then

$$\det([A, i, A_{ii} + t]^u_v) = A_{ij}$$

for some $j \varepsilon \langle \text{order}(A) \rangle$ with $j \neq i$, and paramountcy of A implies the desired inequality.

Thus, assume now that

$$pu = pv < \text{order}(A) - 1.$$

Since $u \varepsilon S_i^{\text{order}(A)}$, $i \varepsilon$ rng $[u * \text{order}(A)]$. Let $\bar{\imath} \varepsilon$ dmn $[u * \text{order}(A)]$. such that

$$[u * \text{order}(A)]_{\bar{\imath}} = i.$$

Suppose now that $i \not\varepsilon$ rng v.

Let $\bar{\jmath} \varepsilon$ dmn $[v * \text{order}(A)]$ such that $[v * \text{order}(A)]_{\bar{\jmath}} = i$. Then

$$\det([A, i, A_{ii} + t]^u_v) = -1^{\bar{\imath}+\bar{\jmath}}\, t \det(([A, i, A_{ii} + t]^u_v)^{\{(1,\bar{\imath})\}}_{\{(1,\bar{\jmath})\}}) + \det(A^u_v).$$

Clearly there exists $(\bar{u}, \bar{v}) \varepsilon G(A)$ such that

$$([A, i, A_{ii} + t]^u_v)^{\{(1,\bar{\imath})\}}_{\{(1,\bar{\jmath})\}} = A^{\bar{u}}_{\bar{v}}.$$

Thus,

$$\det([A, i, A_{ii} + t]^u_v) = -1^{\bar{\imath}+\bar{\jmath}}\, t \det(A^{\bar{u}}_{\bar{v}}) + \det(A^u_v).$$

Note that the above equation is valid when $u = v$ and $\bar{\imath} = \bar{\jmath}$. Hence, the fact that A is paramount implies

$$\begin{aligned}
\det([A, i, A_{ii} + t]^u_u) &- |\det([A, i, A_{ii} + t]^u_v)| \\
&\geq t \det(A^{\bar{u}}_{\bar{u}}) + \det(A^u_u) - (t\,|\det(A^{\bar{u}}_{\bar{v}})| + |\det(A^u_v)|) \\
&= t(\det(A^{\bar{u}}_{\bar{u}}) - |\det(A^{\bar{u}}_{\bar{v}})|) + (\det(A^u_v) - |\det(A^u_v)|) \geq 0.
\end{aligned}$$

Finally, if $i \varepsilon$ rng v, then

$$\det([A, i, A_{ii} + t]^u_v) = \det(A^u_v),$$

and

$$\begin{aligned}
\det([A, i, A_{ii} + t]^u_u) &- |\det([A, i, A_{ii} + t]^u_v)| \\
&= t \det(A^{\bar{u}}_{\bar{u}}) + (\det(A^u_u) - |\det(A^u_v)|) \geq 0.
\end{aligned}$$

The proof is complete.

10. Definition.

B is a reduction of A if and only if A is paramount, and

$$B = [A, i, \inf\{t \mid [A, i, t] \text{ is paramount}\}]$$

for some $i \varepsilon \langle \text{order}(A) \rangle$ such that

$$A_{ii} > \inf\{t \mid [A, i, t] \text{ is paramount}\}.$$

11. Definition.

P is a reduction sequence of A if and only if A is paramount, and P is such a sequence that

$$P_1 = A$$

and

$$P_i \text{ is a reduction of } P_{i-1} \text{ for } 1 < i \in \operatorname{dmn} P.$$

12. Remark.

It is clear from Definitions 10 and 11 that if P is a reduction sequence of A, then

$$pP \leq \text{order } (A).$$

13. Remark.

Suppose that T is such a sequence that

(i) For $i \in \operatorname{dmn} T$ $T(i)$ is a sequence to M.
(ii) $T(1)$ is a reduction sequence of A.
(iii) $(T(i))_{pT(i)}$ is non singular for each $i \in \operatorname{dmn} T$.
(iv) $T(i + 1)$ is a reduction sequence of $[(T(i))_{pT(i)}]^{-1}$ for each $i \in \operatorname{dmn} T$ such that $(i + 1) \in \operatorname{dmn} T$.

Then, clearly, by Jacobi's theorem, whenever $i \in \operatorname{dmn} T$ and $(i + 2) \in \operatorname{dmn} T$,

$$pT(i + 2) < pT(i) \leq \text{order } (A).$$

Thus a sequence T satisfying the above conditions must be finite.

14. Definition.

A is irreducible if and only if A is paramount and there exists no B such that B is a reduction of A.

15. Definition.

T reduces A completely if and only if A is paramount and T is such a finite sequence that

(i) For $i \in \operatorname{dmn} T$ $T(i)$ is a sequence to M.
(ii) $T(1)$ is a reduction sequence of A.
(iii) If $1 < pT$ then $(T(i))_{pT(i)}$ is non singular for $i \in \langle pT - 1 \rangle$.
(iv) If $1 < pT$ then $T(i + 1)$ is a reduction sequence of $[(T(i))_{pT(i)}]^{-1}$ for $i \in \langle pT - 1 \rangle$.
(v) $(T(pT)_{pT(pT)}$ is irreducible and singular, or, $1 < pT$, and $(T(pT))_{pT(pT)}$ and $(T(pT - 1))_{pT(pT-1)}$ are both irreducible.

16. Remark.

If T reduces A completely, then T is maximal in the following precise sense.

Theorem.

Let T reduce A completely and let U reduce A completely such that $T \subset U$. Then $T = U$.

17. Definitions.

(i) If T reduces A completely,

$$n(T, A) = \sum_{i \in \operatorname{dmn} T} pT(i).$$

(ii) If A is paramount and A is not irreducible,

$$n(A) = \sup \{n(T, A) \mid T \text{ reduces } A \text{ completely}\}.$$

If A is paramount and irreducible $n(A) = 0$.

18. Remark.

It is not known whether the following statement is true:
if T reduces A completely and U reduces A completely, then

$$n(T, A) = n(U, A).$$

19. Remark.

If A is paramount, the number $n(A)$ defined in 17 *(ii)* may be relevant to the problem of the realization of A as either the open-circuit impedance matrix or the short-circuit admittance matrix of a resistive network. This topic is discussed in the Introduction. Clearly, if 18 is true, the computation of $n(A)$ will be greatly simplified, and thus determination of the validity of 18 may be of help in the ultimate solution of the realization problem.

20. Acknowledgment.

The research reported in this paper was stimulated by a suggestion of Prof. R. M. Foster of the Polytechnic Institute of Brooklyn. In particular, Prof. Foster suggested that his observation noted in the Footnote cited above might be extended to matrices of higher order.

References

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BOOK REVIEWS

(*Continued from p. 244*)

of a competitive economy, in particular, the convergence over time of the price vector to an equilibrium. For the latter problem a key condition is gross substitutability of commodities, which in mathematical terms says that a certain matrix is of Metzler type (i.e., all nondiagonal elements are nonnegative.)

Volume II is devoted to continuous games, in particular polynomial games, games with convex or bell-shaped kernels, games of timing (duels) and poker models. The main emphasis is on finding the specific structure of the optimal strategies for each player. The art of doing this is somewhat like that of solving differential equations. In each case one must judiciously choose a method which exploits the special features of the kernel; and quite often a good initial guess about the form of the solution is also needed. The methods involve, for example, the geometry of moment spaces, integral equations with positive kernels, and the Neyman-Pearson lemma.

These books are intended partly as textbooks for graduate students and mature undergraduates. They ought to have an important influence on the coming generation of economists and operations researchers. Numerous good problems have been included. In addition the reader is sometimes expected to fill in not entirely obvious details as he goes along, e.g., (p. 123, Vol. I) the fact that a linear function which is bounded above on a closed, unbounded, convex polyhedron P attains its maximum in P. There are Appendixes (worth reading for their own sakes) dealing with vector spaces, matrices, convex sets, and convex functions, to fill possible gaps in the reader's background in these subjects. A knowledge of the rudiments of general topology and real function theory will prove helpful, especially in Volume II. The two volumes are independent of each other; in fact, both have the same first chapter (about matrix games) and the same Appendixes.

Wendell H. Fleming

Hydrodynamics. By D. H. Wilson. St. Martin's Press, New York, 1959. viii + 149 pp. \$5.50.

The purpose of this book is best described by quoting a paragraph from its preface.

"This book is intended primarily for students specializing in mathematics or theoretical physics but will also be found useful by General Degree and Engineering students. Its object is to provide a brief and concise introduction to 'classical' hydrodynamics while at the same time providing the basis and acting as a starting point for the study of the many modern developments in fluid mechanics."

The author suceeds admirably in reaching his objectives. He uses vector calculus to derive, in a very lucid manner, the equations of fluid mechanics in Chapter I. His representation of the basic principles can be recommended for perusal to students who value a fortunate combination of physics and rigorous mathematics. Chapter II proceeds, in conventional fashion, with a study of two dimensional motion of the inviscid incompressible fluid. Chapter III consists of a careful study of vortex motion. Chapters IV and V are devoted to the conformal mapping technique for two dimensional potential flows and the stream function technique for axisymmetrical potential flows respectively. The book closes with a study of viscous motion in Chapter VI. It contains 147 pages.

The author is to be commended for this clear and concise presentation. The book is a worthwile addition to the library of any student of fluid mechanics.

P. F. Maeder

THE UNIQUENESS QUESTION FOR WAVES AGAINST AN OVERHANGING CLIFF*

BY

R. SHERMAN LEHMAN

1. Introduction. For the problem of three-dimensional waves over sloping beaches Roseau [11] and Peters [10] have found explicit solutions for all slope angles. These solutions are in the form of integrals in the complex plane resembling those obtained in the inversion of the Laplace transform. It is not, however, clear that *all* solutions satisfying appropriate conditions can be represented in this way, and indeed in general the uniqueness question is still open. The purpose of the present paper is to study the uniqueness question in a particular case.

The problem of surface waves over sloping beaches when treated by the linearized theory leads to the question of determining a velocity potential $\Phi(x, y, z, t)$ which is a solution of Laplace's equation in the space variables x, y, and z and satisfies two different boundary conditions on different parts of the boundary. Suppose the z-axis is taken along the shore, the y-axis is directed vertically upward with the free surface at $y = 0$, and the x-axis is directed outward from the shore. Then on the free surface $y = 0$, $x > 0$

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial y} = 0,$$

where g is the acceleration due to gravity. On the bottom the normal derivative $\partial\Phi/\partial n = 0$. For a uniformly sloping beach with angle $\pi\alpha$ between the surface and bottom, the bottom is given by the equation $\theta = -\pi\alpha$, when $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$. If $\alpha = \frac{1}{2}$ the "bottom" becomes a vertical cliff, while for $\frac{1}{2} < \alpha < 1$ the "bottom" becomes an overhanging cliff. Our considerations here will be primarily concerned with the case $\alpha = 3/4$.

Restricting consideration to motion which is simple harmonic in the time t and the space variable z, one seeks solutions of the form

$$\Phi(x, y, z, t) = \varphi(x, y)\cos kz \cos(\sigma t + \beta).$$

Progressive waves having crests which at infinity make an arbitrary angle with the shore line can be constructed by taking linear combinations of standing waves of this type.

With an appropriate choice of units of length and time so that $\sigma^2/g = 1$ one obtains the following mixed boundary value problem: Find the solutions of the equation

$$\Delta\varphi(x, y) - k^2\varphi(x, y) = 0 \tag{1.1}$$

in the sector $-\pi\alpha \leq \theta \leq 0$, satisfying

$$\frac{\partial\varphi}{\partial y} - \varphi = 0 \quad \text{for} \quad \theta = 0, \tag{1.2}$$

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and

$$\frac{\partial \varphi}{\partial n} = 0 \qquad \text{for} \quad \theta = -\pi\alpha. \tag{1.3}$$

Roseau [11] and Peters [10] have found solutions for this problem for $k < 1$ and all angles $\pi\alpha \leq \pi$. Roseau [12] has also studied solutions for angles $\pi\alpha < \pi/2$ when $k > 1$. Some of these solutions, which include the Stokes edge waves, have been considered also by Ursell [15]. The question of determining all solutions of the boundary value problem satisfying appropriate conditions at infinity and at the origin has been studied by Stoker [13] and Weinstein [17] for $0 < k < 1$ in the case $\alpha = 1/2$. (See also [14, p. 84].) A proof of uniqueness for $\alpha = 1/4$ can be constructed by using Roseau's special investigation of that case [11, pp. 27–30] together with Stoker's and Weinstein's methods. In this paper we shall use similar methods to handle the case $\alpha = 3/4$.

For $k = 0$ the waves are two-dimensional. In this case solutions for all angles were found earlier by Isaacson [4]. Stoker [13] has treated the uniqueness question for $\alpha = 1/2n$. Brillouët [1] has studied the uniqueness question for $\alpha = p/2q$ following the line of Lewy's original investigation [6]. A uniqueness proof for all $\alpha \leq 1$ for $k = 0$ can be constructed using the methods and results employed by Lewy in [8].

The present investigation was begun with the aim of finding out the true situation with regard to uniqueness for $k > 0$. There is no possibility of answering the question for all angles by the method used here, and even the case of all angles $\pi p/2q$, which in principle probably could be handled by the methods employed here, seems uninvitingly complex. One can hope, however, that knowledge of a few special cases may help in guiding a general consideration using methods similar to those employed by Lewy [7, 8, 9] and the author [5].

2. Reduction to a simpler boundary value problem. We wish to determine for $\alpha = 3/4$ all solutions of the boundary value problem (1.1)–(1.3) which have continuous second derivatives in the sector $-3\pi/4 \leq \theta \leq 0$, $r > 0$ and satisfy two additional boundedness conditions. At infinity we shall require that

$$|\varphi| + \left|\frac{\partial \varphi}{\partial x}\right| + \left|\frac{\partial \varphi}{\partial y}\right| < M \quad \text{for} \quad r > R_0\,, \tag{2.1}$$

where M and R_0 are some positive constants. Near the origin we shall assume that

$$|\varphi| + \left|\frac{\partial \varphi}{\partial x}\right| + \left|\frac{\partial \varphi}{\partial y}\right| < Cr^{-7/3+\epsilon}, \tag{2.2}$$

where C and ϵ are positive constants. Observe that (2.2) allows φ to have a logarithmic singularity at the origin but excludes solutions which behave like $r^{-4/3}$ at the origin.

Because the normal derivative of φ vanishes on $\theta = -3\pi/4$, the function φ can be extended by reflection to the sector $-3\pi/2 \leq \theta \leq 0$. In the larger sector φ will again satisfy the conditions (2.1) and (2.2). Similarly, since $\partial\varphi/\partial y - \varphi$ vanishes on $\theta = 0$, it can be extended by reflection to the sector $-3\pi/4 \leq \theta \leq 3\pi/2$. The resulting function $\partial\varphi/\partial y - \varphi$ is multiple-valued when considered as a function of x and y, but it can be considered as a single-valued function of r and θ. It satisfies the following two relations uniformly in θ:

$$\begin{aligned} \frac{\partial\varphi}{\partial y} - \varphi &= O(1) \qquad \text{for} \quad r \to \infty, \qquad -\tfrac{3}{2}\pi \le \theta \le \tfrac{3}{2}\pi \\ \frac{\partial\varphi}{\partial y} - \varphi &= O(r^{-7/3+\epsilon}) \quad \text{for} \quad r \to 0, \qquad -\tfrac{3}{2}\pi \le \theta \le \tfrac{3}{2}\pi. \end{aligned} \tag{2.3}$$

Since the function φ is symmetric with respect to the line $\theta = -3\pi/4$ and $\partial\varphi/\partial y - \varphi = 0$ for $\theta = 0$, we conclude that $\partial\varphi/\partial x - \varphi = 0$ for $\theta = -3\pi/2$. Define

$$\psi(x, y) = \left(\frac{\partial}{\partial y} - 1\right)\left(\frac{\partial}{\partial x} - 1\right)\varphi(x, y). \tag{2.4}$$

The function ψ is then a solution of the differential equation $\Delta\psi - k^2\psi = 0$ for $-3\pi/2 \le \theta \le 3\pi/2$ because the differential operators involved have constant coefficients. Also $\psi = 0$ for $\theta = 0$ and for $\theta = -3\pi/2$. Thus ψ is a solution of a much simpler boundary value problem.

The method we employ is first to solve this boundary value problem for ψ and then use (2.4) to determine φ by the solution of ordinary differential equations. Finally we must verify that the φ obtained is a solution of (1.1)–(1.3) and satisfies the boundedness conditions (2.1) and (2.2).

It is possible for all $\alpha = p/2q$ with p odd to perform a similar reduction to a problem with vanishing boundary values for a sector with angle $2\pi\alpha$. This simpler boundary value problem can be solved, (see Brillouët [1]) but since for larger p and q the differential operator used in the reduction is more complicated, the determination of φ from the solution of the simpler boundary value problem presents difficulties.

We first use (2.1) and (2.2) together with the following lemma to derive bounds for ψ at the origin and at infinity.

Lemma 1. *Let R_1 be an arbitrary positive number. Let u be any solution of $\Delta u - k^2 u = 0$ existing in a circle of radius $d < R_1$ about (x_0, y_0) and let M_1 be the supremum of u inside this circle. Then*

$$\left|\frac{\partial u}{\partial x}(x_0, y_0)\right| + \left|\frac{\partial u}{\partial y}(x_0, y_0)\right| < L\frac{M}{d},$$

where L is a constant dependent only upon the differential equation and the number R_1.

This lemma is a special case of known theorems. In particular it is a special case of results of Gevrey [3, p. 148] and it also follows from the Schauder interior estimates as stated by Douglis and Nirenberg [2].

We now use the lemma together with (2.3) to obtain estimates for ψ at infinity and at the origin in the sector $-3\pi/4 \le \theta \le 3\pi/4$. For the estimate at infinity we use a circle of radius 1 about the point (x, y). For the estimate at the origin we use a circle of radius $r/2$ about the point (x, y). We find that the following estimates hold uniformly in θ:

$$\begin{aligned} \psi(x, y) &= O(1) \qquad \text{for} \quad r \to \infty, \qquad -\tfrac{3}{4}\pi \le \theta \le \tfrac{3}{4}\pi, \\ \psi(x, y) &= O(r^{-10/3+\epsilon}) \quad \text{for} \quad r \to 0, \qquad -\tfrac{3}{4}\pi \le \theta \le \tfrac{3}{4}\pi. \end{aligned} \tag{2.5}$$

In the following discussion it will sometimes be convenient to consider ψ as a function of r and θ. To avoid possible confusion we set

$$\Psi(r, \theta) = \psi(x, y).$$

Since φ is symmetric with respect to the line $\theta = -3\pi/4$, the function ψ in view of (2.4) also must be symmetric with respect to $\theta = -3\pi/4$. In other words

$$\Psi(r, \theta) = \Psi(r, -\tfrac{3}{2}\pi - \theta). \tag{2.6}$$

By reflection across the line $\theta = 0$ we obtain

$$\Psi(r, \theta) = -\Psi(r, -\theta). \tag{2.7}$$

Combined with (2.6) this yields the information that Ψ is a periodic function of θ with period 3π. For any fixed value of r the function Ψ can therefore be expanded in a convergent Fourier series which because of (2.6) and (2.7) must have the form

$$\Psi(r, \theta) = \sum_{n=0}^{\infty} c_n(r) \sin \tfrac{2}{3}(2n + 1)\theta, \tag{2.8}$$

where the coefficients $c_n(r)$ are given by

$$c_n(r) = \frac{8}{3\pi} \int_{-3\pi/4}^{0} \Psi(r, \theta) \sin \tfrac{2}{3}(2n + 1)\theta \, d\theta. \tag{2.9}$$

The function $\Psi(r, \theta)$ satisfies the partial differential equation

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} - k^2 \Psi = 0.$$

Differentiating (2.9) and using this differential equation, we obtain

$$c_n''(r) + \frac{1}{r} c_n'(r) - \left[k^2 + \left\{\frac{2}{3r}(2n + 1)\right\}^2\right] c_n(r)$$
$$= -\frac{2}{3\pi r^2} \int_{-3\pi/4}^{0} \left[\frac{\partial^2 \Psi}{\partial \theta^2} + (\tfrac{2}{3}(2n + 1))^2 \Psi\right] \sin \tfrac{2}{3}(2n + 1)\theta \, d\theta.$$

The right side of this equation is equal to zero, as can be seen by integrating the first term of the integrand by parts twice and using the fact that $\Psi = 0$ for $\theta = 0$ and $\partial\Psi/\partial\theta = 0$ for $\theta = -3\pi/4$. It follows that the function $c_n(r)$ is a linear combination of modified Bessel functions of order $2(2n + 1)/3$ that is

$$c_n(r) = A_n I_{2(2n+1)/3}(kr) + B_n K_{2(2n+1)/3}(kr), \tag{2.10}$$

where A_n and B_n are real constants.

Next we use the estimates (2.5) to prove that all of these constants except B_0 and B_1 are equal to zero. For $\rho \to \infty$ over positive values the following asymptotic relations hold (see [16, p. 202]):

$$\begin{aligned} I_\nu(\rho) &\sim (2\pi\rho)^{-1/2} e^{\rho}, \\ K_\nu(\rho) &\sim \left(\frac{\pi}{2\rho}\right)^{1/2} e^{-\rho}. \end{aligned} \tag{2.11}$$

Inserting the first estimate of (2.5) into (2.9) and letting $r \to \infty$, we conclude that $A_n = 0$ for $n = 0, 1, 2, \cdots$. Also for $\rho \to 0$ over positive values

$$K_\nu(\rho) = O(\rho^{-\nu}), \tag{2.12}$$

when $\nu > 0$ [16, p.77–78]. Hence the second estimate of (2.5) when combined with

(2.9) allows us to conclude that $B_n = 0$ for $n = 2, 3, 4, \cdots$. Thus we find

$$\psi(x, y) = B_0 K_{2/3}(kr) \sin \tfrac{2}{3}\theta + B_1 K_2(kr) \sin 2\theta, \tag{2.13}$$

where B_0 and B_1 are real constants.

3. Solution of ordinary differential equations. In Sec. 2 we proved that if $\varphi(x, y)$ is a solution of the boundary value problem (1.1)–(1.3) satisfying the boundedness conditions (2.1) and (2.2), then the corresponding function $\psi\ (x, y)$ determined by (2.4) must have the form (2.13). In this section we shall prove that corresponding to each function ψ of the form (2.13) there is at most one function φ satisfying (1.1)–(1.3), (2.1) and (2.2), thus establishing that for each $k > 0$ there is at most a two-parameter family of solutions of our problem. The method employed in this section is essentially integrating ordinary differential equations.

We shall need the following lemma.

Lemma 2. Suppose $u(x, y)$ is a solution of the differential equation $\Delta u - k^2 u = 0$ in the strip $a < x < b$, $-\infty < y < \infty$. Suppose also that there is a constant C_2 such that

$$|u| + \left|\frac{\partial^2 u}{\partial x^2}\right| + \left|\frac{\partial u}{\partial y}\right| < C_2 \tag{3.1}$$

for $y > y_0$, $a < x < b$. Then the function

$$U(x, y) = e^y \int_{+\infty}^{y} e^{-t} u(x, t)\, dt$$

is a solution of the same differential equation in the same strip.

Proof. By differentiating we obtain

$$\Delta U - k^2 U = e^y \int_{\infty}^{y} e^{-t}\left[\frac{\partial^2 u(x, t)}{\partial x^2} + (1 - k^2) u(x, t)\right] dt + u(x, y) + \frac{\partial}{\partial y} u(x, y),$$

as is easily justified in view of (3.1). Transforming the integrand by using the differential equation and then integrating by parts, we obtain

$$\Delta U - k^2 U = e^y \int_{\infty}^{y} e^{-t}\left(u - \frac{\partial^2 u}{\partial t^2}\right) dt + u(x, y) + \frac{\partial}{\partial y} u(x, y)$$

$$= e^y \int_{\infty}^{y} e^{-t}\left(\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t}\right) dt = 0.$$

The inequality (3.1) insures that the boundary terms at the limit $+\infty$ vanish.

Now let us make a cut along the positive y-axis of the x, y plane and restrict θ to the range $-3\pi/2 < \theta < \pi/2$. In this way for given values of B_0 and B_1 in (2.13) we obtain a single-valued function $\psi(x, y)$ defined in the entire cut plane. Let

$$v(x, y) = \left(\frac{\partial}{\partial x} - 1\right)\varphi(x, y) \tag{3.2}$$

so that v is a solution of the equation

$$\left(\frac{\partial}{\partial y} - 1\right)v(x, y) = \psi(x, y). \tag{3.3}$$

Integrating we find

$$v(x, y) = e^y g(x) + e^y \int_\infty^y e^{-t}\psi(x, t)\, dt \equiv e^y g(x) + v^*(x, y), \tag{3.4}$$

where g is a function of x alone. By Lemma 2 $v^*(x, y)$ is a solution of the differential equation (1.1) for $-\infty < x < 0$, $-\infty < y < \infty$ and also it is a solution for $0 < x < \infty$, $-\infty < y < \infty$. As we shall see, $v^*(x, y)$ is discontinuous on the entire y-axis. Since the function v is also a solution of (1.1), $e^y g(x)$ must be a solution, i.e.

$$g''(x) + (1 - k^2)g(x) = 0 \tag{3.5}$$

for the intervals $-\infty < x < 0$ and $0 < x < \infty$. We shall denote by $g_1(x)$ and $g_2(x)$ the analytic functions of x coinciding with g for $x < 0$ and $x > 0$, respectively.

By (2.11) $\psi(x, t)$ is exponentially small for $x^2 + t^2$ large. Consequently for $x \to +\infty$ with y fixed

$$v(x, y) = e^y g_2(x) + o(1).$$

Also by (3.2) and (2.1) we know $v(x, y) = O(1)$ for $x \to +\infty$, $y \le 0$. When $k < 1$, this does not impose any additional condition on $g_2(x)$; but if $k > 1$ among the solutions of (3.5), $a_1 e^{k_1 x} + a_2 e^{-k_1 x}$ with $k_1 = (k^2 - 1)^{1/2}$, we must reject all those for which $a_1 \neq 0$. Hence we obtain the additional condition

$$g_2'(0) = -k_1 g_2(0), \qquad k_1 = (k^2 - 1)^{1/2}, \tag{3.6}$$

which must be satisfied in order to obtain a φ satisfying (2.1). Similarly for $k = 1$ we must have $g_2'(0) = 0$. Thus (3.6) must be imposed for $k \ge 1$.

Next, solving (3.2) for $\varphi(x, y)$ in the lower half plane, we obtain

$$\varphi(x, y) = e^x h(y) + e^x \int_\infty^x e^{-t} v(t, y)\, dt \tag{3.7}$$

By an argument using Lemma 2 with x and y interchanged one can show that the second term on the right hand side is a solution of the differential equation (1.1) for $y < 0$. It follows that for $y < 0$

$$h''(y) + (1 - k^2)h(y) = 0.$$

By (1.2) we must have $\partial\varphi/\partial y - \varphi = 0$ for $y = 0$, $x > 0$. For all $y < 0$ we have

$$\begin{aligned} \frac{\partial\varphi}{\partial y} - \varphi &= e^x(h'(y) - h(y)) + e^x \int_\infty^x e^{-t}\left(\frac{\partial}{\partial y} - 1\right)v(t, y)\, dt \\ &= e^x(h'(y) - h(y)) + e^x \int_\infty^x e^{-t}\psi(t, y)\, dt. \end{aligned} \tag{3.8}$$

Keeping x fixed > 0 and taking the limit for $y \to 0^-$, we conclude that we must have

$$h'(0) - h(0) = 0.$$

Since, as we have seen, $v(t, y) = O(1)$ for $t \to +\infty$, we obtain from (3.7)

$$\varphi(x, y) = e^x h(y) + O(1)$$

for $x \to +\infty$ with y fixed. To have φ bounded as required by (2.1) we must have $h'(0) = h(0) = 0$, and hence by the differential equation $h(y) = 0$ for all y.

The boundary condition (1.3) also must be satisfied. For $y = x$, $x < 0$ we have, in view of (3.2) and (3.8),

$$\begin{aligned} 2^{1/2}\frac{\partial\varphi}{\partial n} &= \frac{\partial\varphi}{\partial x} - \frac{\partial\varphi}{\partial y} \\ &= [\varphi(x, y) + v(x, y)] - \left[e^x \int_\infty^x e^{-t}\psi(t, y)\,dt + \varphi(x, y)\right] \\ &= v(x, y) - e^x \int_\infty^x e^{-t}\psi(y, t)\,dt \quad (\text{since} \quad \psi(t, y) = \psi(y, t)) \\ &= g(x)e^x, \quad \text{by} \quad (3.4). \end{aligned} \tag{3.9}$$

Consequently we must have $g_1(x) = 0$ for all x.

Since v and $\partial v/\partial x$ must be continuous across the negative y-axis, we must have for $y < 0$

$$\begin{aligned} g_2(0) &= \lim_{x\to 0^-}\int_\infty^y e^{-t}\psi(x, t)\,dt - \lim_{x\to 0^+}\int_\infty^y e^{-t}\psi(x, t)\,dt, \\ g_2'(0) &= \lim_{x\to 0^-}\int_\infty^y e^{-t}\frac{\partial\psi}{\partial x}\,dt - \lim_{x\to 0^+}\int_\infty^y e^{-t}\frac{\partial\psi}{\partial x}\,dt. \end{aligned} \tag{3.10}$$

The function $g_2(x)$ is uniquely determined by $g_2(0)$ and $g_2'(0)$ since it is a solution of the differential equation (3.5). In the next section we shall establish that these limits exist for every function ψ of the form (2.13) and are independent of y. Without further work, however, we can conclude that if there is a solution corresponding to ψ, it is given by the formula

$$\varphi(x, y) = e^{x+y}\int_\infty^x e^{-t}g(t)\,dt + e^{x+y}\int_\infty^x\int_\infty^y e^{-t-\tau}\psi(\tau, t)\,dt\,d\tau, \tag{3.11}$$

where

$$g(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ g_2(x) & \text{if} \quad x > 0. \end{cases}$$

Observe that by (3.10) and (3.11) φ is uniquely determined by ψ and also that the dependence is linear. From (2.13) we see that for each $k > 0$ there are at most two linearly independent solutions of the boundary value problem satisfying the conditions (2.1) and (2.2). In fact, for $k < 1$ Roseau [11] and Peters [10] have found two linearly independent solutions, one of which is finite at the origin, and the other of which has a logarithmic singularity there. If one used their work, to complete the proof that for each ψ the formula (3.11) represents a solution of our problem, one would only need to verify that their solutions satisfy the conditions (2.1) and (2.2). However, we shall not do this, but instead we shall give a proof which is independent of their work. At the same time we shall complete the study for $k \geq 1$, where it turns out that (because of (3.6)) there is only one linearly independent solution of the problem and it has a logarithmic singularity at the origin.

To summarize, we have proved that for each ψ of the form (2.13), if the limits (3.10) exist, then the function $\varphi(x, y)$ given by (3.11) is a solution of the boundary value problem (1.1)–(1.3). In the next section we show that the limits (3.10) exist and at the

same time evaluate them explicitly by using Bessel function formulas. Then in the last section we study the behavior at infinity and at the origin of $\varphi(x, y)$ as given by (3.11) and in doing so verify that it satisfies the conditions (2.1) and (2.2).

4. Evaluation of some limits. We shall use the formula (see [16 p. 388])

$$\int_0^\infty e^{-t \cosh \beta} K_\nu(t)\, dt = \frac{\pi}{\sin \nu\pi} \cdot \frac{\sinh \nu\beta}{\sinh \beta}$$

valid for $0 < \nu < 1$, Re $\cosh \beta > -1$, $-\pi/2 \leq \text{Im} \cosh \beta \leq \pi/2$. It follows that

$$\int_0^\infty e^{-t} K_\nu(kt)\, dt = \frac{\pi}{k \sin \nu\pi} \cdot \frac{\sinh \nu\beta}{\sinh \beta} \tag{4.1}$$

for $0 < \nu < 1$, $0 < k < \infty$, $\cosh \beta = 1/k$ with $0 \leq \text{Im} \cosh \beta \leq \pi/2$. More precisely, if $k = 1$, then $\beta = 0$; and the ratio $\sinh \nu\beta/\sinh \beta$ is defined to have the value ν. For $k > 1$, β is complex and therefore sometimes it may be more convenient to use the alternative expression

$$\frac{\pi}{k \sin \nu\pi} \cdot \frac{\sin \nu\gamma}{\sin \gamma}$$

with $\cos \gamma = 1/k$, $0 \leq \gamma < \pi/2$.

To abbreviate, we introduce a limit operator δ defined by the equation

$$\delta f = \lim_{x \to 0^+} f(x, y) - \lim_{x \to 0^-} f(x, y)$$

for any function f for which the limit exists. Then (3.10) becomes

$$\begin{aligned} g_2(0) &= \delta \int_y^\infty e^{-t} \psi(x, t)\, dt, \\ g_2'(0) &= \delta \int_y^\infty e^{-t} \frac{\partial}{\partial x} \psi(x, t)\, dt \end{aligned} \tag{4.2}$$

provided the limits exist. We shall prove that these limits do exist for every $y < 0$ and are independent of y. Throughout this section we shall assume that $y < 0$.

We investigate separately the contributions due to each term of ψ as given by (2.13). In each case in the following discussion where we obtain a limit of an integral by passing to the limit under the integral sign, this can be justified by the Lebesgue dominated convergence theorem. First we have

$$\begin{aligned} &\delta \int_y^\infty e^{-t} K_{2/3}(k[t^2 + x^2]^{1/2}) \sin \tfrac{2}{3}\theta\, dt \\ &\qquad = \int_0^\infty e^{-t} K_{2/3}(kt)\, [\sin (\pi/3) - \sin (-\pi)]\, dt = \frac{\pi}{k} \frac{\sinh (2\beta/3)}{\sinh \beta}. \end{aligned}$$

Using the formula

$$\frac{\partial^2}{\partial x\, \partial y} K_0(kr) = \frac{k^2}{2} K_2(kr) \sin 2\theta$$

and integrating by parts, we find

$$L_1 \equiv \delta \int_y^\infty e^{-t} K_2(k[t^2 + x^2]^{1/2}) \sin 2\theta\, dt$$

$$= \delta \int_y^\infty \frac{2}{k^2} e^{-t} \frac{\partial^2}{\partial t\, \partial x} K_0(k[t^2 + x^2]^{1/2})\, dt$$

$$= \delta \int_y^\infty \frac{2}{k^2} e^{-t} \frac{\partial}{\partial x} K_0(k[t^2 + x^2]^{1/2})\, dt.$$

Also (see [16, pp. 79, 80])

$$\frac{\partial}{\partial x} K_0(kr) = -\frac{kx}{r} K_1(kr) = -\frac{x}{r^2} + xp(r),$$

where there is a constant C such that for $0 < r < \infty$

$$| p(r) | < C(1 + | \log r |).$$

Consequently

$$L_1 = -\delta \int_y^\infty \frac{2}{k^2} e^{-t} \frac{x}{x^2 + t^2}\, dt + \delta \int_y^\infty \frac{2}{k^2} e^{-t} xp([t^2 + x^2]^{1/2}) dt.$$

The second term is 0 since the limit of the integrand is 0 for $x \to 0$. The first term can be evaluated by introducing $\tau = t/x$ as a new variable of integration. Since $y < 0$, we obtain

$$L_1 = -\delta \int_y^\infty \frac{2}{k^2} e^{-t} \frac{x}{x^2 + t^2}\, dt = -\frac{4}{k^2} \lim_{x \to 0+} \int_{y/x}^\infty \frac{e^{-\tau x}}{1 + \tau^2}\, d\tau$$

$$= -\frac{4}{k^2} \int_{-\infty}^\infty \frac{d\tau}{1 + \tau^2} = -\frac{4\pi}{k^2}.$$

From the formula (see [16, p. 79])

$$zK_\nu'(z) + \nu K_\nu(z) = -zK_{\nu-1}(z)$$

one can derive the following identity:

$$\frac{\partial}{\partial x} (K_\nu(kr) \sin \nu\theta) = \frac{\partial}{\partial y} (K_\nu(kr) \cos \nu\theta) - kK_{\nu-1}(kr) \sin (\nu - 1)\theta.$$

Using this identity and the fact that $K_{-\nu}(z) = K_\nu(z)$, we obtain

$$\delta \int_y^\infty e^{-t} \frac{\partial}{\partial x} \{K_{2/3}(k[t^2 + x^2]^{1/2}) \sin \tfrac{2}{3}\theta\}\, dt$$

$$= \delta \int_y^\infty e^{-t} \frac{\partial}{\partial t} \{K_{2/3}(k[t^2 + x^2]^{1/2}) \cos \tfrac{2}{3}\theta\}\, dt$$

$$+ \delta \int_y^\infty ke^{-t} K_{1/3}(k[t^2 + x^2]^{1/2}) \sin \tfrac{1}{3}\theta\, dt \equiv L_2 + L_3\,.$$

The limit L_2 can be evaluated by first integrating by parts and then passing to the limit under the integral sign to obtain by (4.1)

$$L_2 = \int_0^\infty e^{-t} K_{2/3}(kt)[\cos (\pi/3) - \cos (-\pi)]\, dt = \frac{3^{1/2}\pi}{k} \frac{\sinh \frac{2}{3}\beta}{\sinh \beta}.$$

Also

$$L_3 = k \int_0^\infty e^{-t} K_{1/3}(k\sqrt{t^2 + x^2})[\sin (\pi/6) - \sin (-\pi/2)]\, dt = 3^{1/2}\pi \frac{\sinh (\beta/3)}{\sinh \beta}.$$

We have

$$L_4 \equiv \delta \int_y^\infty e^{-t} \frac{\partial}{\partial x} (K_2(k[t^2 + x^2]^{1/2}) \sin 2\theta)\, dt$$
$$= \delta \int_y^\infty \frac{2e^{-t}}{k^2} \frac{\partial^3}{\partial t\, \partial x^2} (K_0(k[t^2 + x^2]^{1/2}))\, dt$$
$$= \delta \int_y^\infty \frac{2}{k^2} e^{-t} \frac{\partial^2}{\partial x^2} K_0(k[t^2 + x^2]^{1/2})\, dt.$$

Since $K_0(kr)$ is a solution of $\Delta K_0 - k^2 K_0 = 0$, we obtain

$$L_4 = \delta \int_y^\infty 2e^{-t} K_0(k[t^2 + x^2]^{1/2})\, dt - \delta \int_y^\infty \frac{2e^{-t}}{k^2} \frac{\partial^2}{\partial t^2} K_0(k[t^2 + x^2]^{1/2})\, dt$$
$$= \delta \int_y^\infty \left(2 - \frac{2}{k^2}\right) e^{-t} K_0(k[t^2 + x^2]^{1/2})\, dt = 0,$$

since the limit of the integrand is the same for $x \to 0^+$ as it is for $x \to 0^-$.

Thus

$$\begin{aligned} g_2(0) &= B_0 \frac{\pi}{k} \frac{\sinh (2\beta/3)}{\sinh \beta} - B_1 \frac{4\pi}{k^2} \\ g_2'(0) &= B_0 \left\{ \frac{3^{1/2}\pi}{k} \frac{\sinh (2\beta/3)}{\sinh \beta} + 3^{1/2}\pi \frac{\sinh (\beta/3)}{\sinh \beta} \right\} \end{aligned} \tag{4.3}$$

where $\cosh \beta = 1/k$, $\operatorname{Re} \beta \geq 0$, $0 \leq \operatorname{Im} \beta \leq \pi/2$. It is easily verified that the coefficients of B_0 in these expressions never vanish.

When combined with (3.6) these formulas establish that for $k \geq 1$, B_0 is determined uniquely if B_1 is given. For example for $k = 1$ we must have $B_0 = 0$. Thus for $k \geq 1$ there is at most one linearly independent solution satisfying the conditions (2.1) and (2.2).

5. Behavior at infinity and at the origin. We now investigate the behavior at infinity and at the origin of $\varphi(x, y)$ as given by (3.11), (2.13), (4.3), and (3.5). We begin by estimating

$$v^*(x, y) = e^y \int_\infty^y e^{-t} \psi(x, t)\, dt \tag{5.1}$$

(see (3.4)). If ψ is any one of the functions given by (2.13) and p is an arbitrary positive number, then there is by (2.11) a corresponding number c_1 such that

$$| \psi(x, y) | < c_1 e^{-kr} \quad \text{for} \quad r = (x^2 + y^2)^{1/2} \geq p. \tag{5.2}$$

(It is necessary to omit a neighborhood of the origin because of the singularity of K_ν there). For $y \geq 0$, $x^2 + y^2 \geq p^2$ we obtain

$$| v^*(x, y) | \leq e^y \int_y^\infty c_1 e^{-(k+1)t}\, dt \leq \frac{c_1}{k+1} e^{-ky}. \tag{5.3}$$

For $y < 0$, $x^2 + y^2 \geq p^2$, $x \neq 0$ we write

$$v^*(x, y) = e^y \int_\infty^{-p} e^{-t} \psi(x, t)\, dt + e^y \int_{-p}^y e^{-t} \psi(x, t)\, dt \equiv T_1 + T_2\,. \tag{5.4}$$

By the results of the last section we know the *integral* in T_1 approaches a limit as $x \to 0^+$ and also approaches a limit as $x \to 0^-$. Furthermore by (5.2) the integral in T_1 approaches 0 as $|x| \to \infty$. Hence there is a number c_2 such that $|T_1| < c_2 e^y$.

For the second term T_2 we have by (5.2)

$$| T_2 | < c_1 e^y \left| \int_{-p}^{y} e^{(k-1)t} \, dt \right|,$$

which is less than or equal to

$$\frac{c_1 e^y}{|k - 1|} \{e^{(k-1)y} + e^{-(k-1)p}\}$$

for $k \neq 1$ and is less than or equal to

$$c_1 e^y(|y| + p)$$

for $k = 1$. In either case, if we define

$$\mu = \begin{cases} k & \text{for} \quad k < 1 \\ 1 - \eta & \text{for} \quad k = 1 \\ 1 & \text{for} \quad k > 1 \end{cases}$$

with η an arbitrarily small positive number, then there is a constant c_3 independent of x and y such that $|T_2| < c_3 e^{\mu y}$. Combining this estimate with that for $|T_1|$ and (5.3) we find

$$| v^*(x, y) | < c_4 e^{-\mu |y|} \quad \text{for} \quad x \neq 0, \qquad x^2 + y^2 \geq p^2. \tag{5.5}$$

Since for $y \leq 0$,

$$\varphi(x, y) = e^{x+y} \int_{\infty}^{x} e^{-t} g(t) \, dt + e^x \int_{\infty}^{x} e^{-t} v^*(t, y) \, dt,$$

we obtain immediately

$$\left| \varphi(x, y) - e^{x+y} \int_{\infty}^{x} e^{-t} g(t) \, dt \right| < c_4 e^{\mu y}.$$

For $k < 1$ with $k_2 = (1 - k^2)^{1/2}$ we have

$$g_2(t) = g_2(0) \cos k_2 t + k_2^{-1} g_2'(0) \sin k_2 t,$$

and hence for $x \geq 0$

$$e^x \int_{\infty}^{x} e^{-t} g(t) \, dt = \frac{1}{(2 - k^2)} \{-(g_2(0) + g_2'(0)) \cos k_2 x + (k_2^2 g_2(0) - g_2'(0)) k_2^{-1} \sin k_2 x\}, \tag{5.7}$$

while for $x < 0$ since $g(t) = 0$ for $t < 0$,

$$e^x \int_{\infty}^{x} e^{-t} g(t) \, dt = e^x \int_{\infty}^{0} e^{-t} g_2(t) \, dt = \frac{-e^x}{(2 - k^2)} (g_2(0) + g_2'(0)). \tag{5.8}$$

If $k \geq 1$, assuming (3.6) is satisfied, we have

$$g_2(t) = g_2(0) e^{-k_1 t},$$

where $k_1 = (k^2 - 1)^{1/2}$. Hence for $x \geq 0$

$$e^x \int_\infty^x e^{-t} g(t)\, dt = \frac{-g_2(0)e^{-k_1 x}}{1 + k_1} \tag{5.9}$$

while for $x < 0$

$$e^x \int_\infty^x e^{-t} g(t)\, dt = \frac{-g_2(0)e^x}{1 + k_1}. \tag{5.10}$$

Thus we see that for $y \leq 0$, $\varphi(x, y)$ is bounded outside a neighborhood of the origin for every function ψ given by (2.13) in case $k < 1$; and the same is true for $k \geq 1$ provided (3.6) holds.

To study the behavior on the free surface $y = 0$ it is desirable to use a better estimate for v^* there. By (5.2) for $x > p$ we have $|\psi(x, t)| < c_1 e^{-kx}$. It follows that $|v^*(x, 0)| < c_1 e^{-kx}$ and hence for $y = 0$ the right side of (5.6) can be replaced by $c_1 e^{-kx}/(k + 1)$. Equation (5.7) then shows that for $k < 1$, $\varphi(x, 0)$ approaches a sinusoidal wave as $x \to +\infty$. Similarly for $k = 1$, $\varphi(x, 0)$ approaches a constant as $x \to +\infty$, the constant being $-g_2(0)$. For $k > 1$, $\varphi(x, 0) = O(e^{-k_1 x})$ as $x \to +\infty$.

To complete the verification that condition (2.1) is satisfied we must verify that $|\partial\varphi/\partial x|$ and $|\partial\varphi/\partial y|$ also are bounded. By (3.2), (3.4), and (3.8) we have

$$\begin{aligned} \frac{\partial\varphi}{\partial x} &= \varphi(x, y) + v(x, y) = \varphi(x, y) + e^y g(x) + v^*(x, y), \\ \frac{\partial\varphi}{\partial y} &= \varphi(x, y) + e^x \int_\infty^x e^{-t}\psi(t, y)\, dt = \varphi(x, y) + v^*(y, x) \end{aligned} \tag{5.11}$$

since $\psi(y, t) = \psi(t, y)$. The boundedness then follows in view of the estimate (5.5) for v^*.

Next we study the behavior of φ near the origin. In view of (2.13), it is convenient to consider separately two different functions ψ,

$$\begin{aligned} \psi_0(x, y) &= K_{2/3}(kr) \sin(2\theta/3), \\ \psi_1(x, y) &= \frac{k^2}{2} K_2(kr) \sin 2\theta = \frac{\partial^2}{\partial x\, \partial y} K_0(kr), \end{aligned}$$

and obtain estimates for the corresponding functions v_0^* and v_1^*.

We have

$$v_0^*(x, y) = e^y \int_\infty^y e^{-t} K_{2/3}(k[x^2 + t^2]^{1/2}) \sin(2\theta/3)\, dt = O(1)$$

for all x and all $y \leq 0$.

On the other hand,

$$\begin{aligned} v_1^*(x, y) &= e^y \int_\infty^y e^{-t} \frac{\partial^2}{\partial t\, \partial x} K_0(k[t^2 + x^2]^{1/2})\, dt \\ &= \frac{\partial}{\partial x} K_0(k[x^2 + y^2]^{1/2}) + e^y \int_\infty^y e^{-t} \frac{\partial}{\partial x} K_0(k[t^2 + x^2]^{1/2})\, dt \\ &= \frac{\partial}{\partial x} K_0(kr) + O(1) \end{aligned}$$

for $x^2 + y^2 \to 0$, $y \leq 0$, as can be seen by the argument used to evaluate L_1 in the previous section. Hence

$$e^x \int_\infty^x e^{-t} v^*(t, y)\, dt = B_1 e^x \int_\infty^x e^{-t} \frac{\partial}{\partial t} K_0(k[t^2 + y^2]^{1/2})\, dt + O(1)$$

$$= K_0(kr) + e^x \int_\infty^x e^{-t} K_0(k[t^2 + y^2]^{1/2})\, dt + O(1)$$

$$= K_0(kr) + O(1)$$

for $x^2 + y^2 \to 0$. Hence we obtain

$$\varphi(x, y) = \frac{2}{k^2} B_1 K_0(kr) + O(1)$$

for $r \to 0$. Thus if $B_1 = 0$ we obtain a solution which is finite at the origin; if $B_1 \neq 0$ the solution will have a logarithmic singularity at the origin.

To complete the verification of the condition (2.2) one uses (5.11) to estimate the derivatives.

In summary, for $k < 1$ there are two linearly independent solutions satisfying the conditions (2.1) and (2.2). One can be taken to be finite at the origin. The other then has a logarithmic singularity there. At infinity for $y = 0$ both of these solutions have a sinusoidal form. On the other hand for $k \geq 1$, there is only one linearly independent solution satisfying the conditions (2.1) and (2.2) and it has a logarithmic singularity at the origin. For $k = 1$ with $y = 0$ this solution approaches a constant for $x \to +\infty$; for $k > 1$ the solution dies out exponentially as $x \to \infty$. In every case the solutions die out exponentially as $y \to -\infty$.

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QUASI-TRIDIAGONAL MATRICES AND TYPE-INSENSITIVE DIFFERENCE EQUATIONS*

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1. Introduction. In solving linear partial differential equations by finite difference methods a boundary value problem is reduced to solving a set of linear equations. In such instances the matrix involved usually takes a special form and consists mainly of zeros. Many of these matrices fall into the class to be considered here which may be called *quasi-tridiagonal* matrices. That is, we consider partitioned matrices of the form

$$Q = \begin{bmatrix} M_1 & E_1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ D_2 & M_2 & E_2 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & D_3 & M_3 & E_3 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & D_{q-1} & M_{q-1} & E_{q-1} \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & D_q & M_q \end{bmatrix} = [D_n\,, M_n\,, E_n]_1^q\,, \qquad (1.1)$$

where the D_n, M_n, E_n are matrices with the same number of rows, E_n, M_{n+1}, D_{n+2} have the same number of columns, and the M_n are square. We propose to solve

$$Qv = g \qquad (1.2)$$

by direct methods.

Various discretizations lead to matrices of this type. That the usual finite-difference approximation of certain boundary problems for the Poisson and the bi-harmonic equation yield matrices of the form (1.1) has been shown by O. Karlqvist [4], A. F. Cornock [2], L. H. Thomas [8] and others [6, 7]. Whereas the usual method of solution of the resulting linear equation is by iteration these authors propose direct methods for solving the above problems.

The processes described here are, in part, extensions of those described by Karlqvist [4] and Cornock [2]. Furthermore it is shown that the finite difference equations obtained from symmetric positive systems, as defined by K. O. Friedrichs [5], also fall into the class of matrices of the form (1.1). These include, in addition to pure elliptic or hyperbolic equations, a certain class of boundary problems for equations of mixed type such as the Tricomi equation. Where iterative methods for such problems seem to present difficulties, even when Q is positive definite and symmetric, the direct methods for solving (1.2) are shown below to be feasible for this larger class of problems.

A criterion is given for the process to apply which is similar to that found for the

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LDU theorem [3]. It is also shown that when the M_n have the same order p, and the D_n are easily invertible, then the process may be reduced to multiplication of matrices and the inversion of one matrix of order p. This last fact was noted by Cornock [2] for the Poisson and bi-harmonic case.

More generally, if for $r = 1, 2, \cdots, k$; $k \leq q$, and for integers q_r such that $1 \leq q_1 < q_2 < \cdots < q_k = q$, $q_0 = 0$, the matrices M_n, for $q_{r-1} < n \leq q_r$, all have the same order p_r then it is shown that the process may be reduced to multiplications and the inversion of k matrices of orders $p_1, \cdots, p_k$.

A code to solve (1.2) has been written by Max Goldstein for the I.B.M.-704 at New York University and this code has been successfully applied to a symmetric positive problem for the Tricomi equation.

This code has also been applied to solving pure elliptic problems to compare the direct method with iterative methods. The direct method was used, for instance to solve the bi-harmonic boundary problem of a simply supported rectangular slab. A comparison of running time would indicate that the direct method is considerably faster than iterative methods for this type of problem.

2. Direct methods. We seek a reduction of Q to the form

$$Q = LU, \tag{2.1}$$

where L and U are square matrices, partitioned in the same manner as Q, of the form

$$L = [C_n, I_n, 0]_1^q, \tag{2.2}$$

$$U = [0, A_n, E_n]_1^q, \tag{2.3}$$

where I_n is a unit matrix of the same order as M_n. We also partition the column vectors v and g in the same manner.

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_q \end{bmatrix}, \qquad g = \begin{bmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_q \end{bmatrix}$$

Comparing right and left hand sides of (2.1) we set, for $1 < n \leq q$,

$$A_1 = M_1, \qquad D_n = C_n A_{n-1}, \qquad M_n = C_n E_{n-1} + A_n .$$

If the A_n are non-singular, the A_n and C_n may be obtained recursively from

$$A_n = M_n - D_n A_{n-1}^{-1} E_{n-1}, \qquad A_1 = M_1 \tag{2.4}$$

$$C_n = D_n A_{n-1}^{-1}, \qquad 1 < n \leq q. \tag{2.5}$$

To solve for v let $Uv = y$, then $Ly = g$; y and v may then be obtained recursively from

$$y_n = g_n - C_n y_{n-1} \qquad 1 < n \leq q, \tag{2.6}$$

where

$$y_1 = g_1, \quad \text{and}$$

$$v_n = A_n^{-1}(y_n - E_n v_{n+1}) \qquad 1 \leq n < q, \tag{2.7}$$

where

$$v_q = A_q^{-1} y_q .$$

We will refer to this recursive method (2.4)-(2.7) as an LU-process. It should be noted that this process requires the inversion of $A_1, A_2, \cdots, A_q$ and their storage for use on the "backward sweep" (2.7).

In certain cases most of these inversions may be avoided. For instance, if all the matrices M_n are of the same order p and the D_n are easily invertible, we may premultiply Q by the quasi-diagonal matrix

$$\begin{bmatrix} I & & & & 0 \\ & D_2^{-1} & & & \\ & & D_3^{-1} & & \\ & & & \ddots & \\ 0 & & & & D_q^{-1} \end{bmatrix}.$$

We may thus assume that $D_n = I$, $2 \le n \le q$. If we let $A_n = H_{n-1}^{-1} H_n$ then the LU-process becomes

$$H_n = H_{n-1} M_n - H_{n-2} E_{n-1} \qquad 2 \le n \le q, \tag{2.8}$$

where $H_0 = I$, $H_1 = M_1$;

$$H_{n-1}(y_n - g_n) = -H_{n-2} y_{n-1}, \qquad 2 \le n \le q, \tag{2.9}$$

where $y_1 = g_1$ and $H_q v_q = H_{q-1} y_q$.

If we let $u_n = H_{n-1} y_n$ then from (2.9)

$$u_n = H_{n-1} g_n - u_{n-1}, \qquad 1 < n \le q, \tag{2.10}$$

where $u_1 = g_1$. To obtain the v_n we go back to the original equation (1.2) whence

$$v_{n-1} = g_n - M_n v_n - E_n v_{n+1}, \qquad 1 \le n < q - 1 \tag{2.11}$$

where $v_{q-1} = g_q - M_q v_q$ and v_q is solved from

$$H_q v_q = u_q . \tag{2.12}$$

We have, in this case, only one matrix H_q of order p to invert.

This recursion (2.8)-(2.12) will be called an H-process. This method may be used, for instance, in solving problems for the Poisson or bi-harmonic equations over a rectangle. In the case of Poisson's equation the q inversions of the LU-process are reduced to solving only one set of p equations where p is the number of mesh points on a horizontal line. This fact was noted for these equations by Cornock [2] in specific examples.

For regions which are made up of rectangles a similar result may be obtained. For instance, in solving Poisson's equation for an L-shaped region made up of two rectangles R_1, R_2 each having p_1, p_2 points on a horizontal mesh line, respectively, only two matrices of order p_1 and p_2 need be inverted.

To treat the general case let us assume that for $1 \le q_1 < q_2 < \cdots < q_k = q$, $q_0 = 0$ the M_n have the same order p_r for $q_{r-1} < n \le q_r$. The matrix Q can then be partitioned again by combining those M_n having the same order. That is, let

$$Q = [D'_n, M'_n, E'_n]_1^k,$$

where

$$D'_{r+1} = \begin{bmatrix} 0 & \cdots & 0 & D_{q_r+1} \\ & & & 0 \\ & & & \vdots \\ 0 & & & 0 \end{bmatrix}, \qquad 1 \leq r < k,$$

$$M'_{r+1} = [D_n, M_n, E_n]_{q_r+1}^{q_{r+1}}, \qquad 0 \leq r < k$$

$$E'_{r+1} = \begin{bmatrix} 0 & & & 0 \\ \vdots & & & \\ 0 & & & \\ E_{q_{r+1}} & 0 & \cdots & 0 \end{bmatrix}, \qquad 0 \leq r < k-1.$$

The LU-process can be performed for this new form for Q. Let g'_r, y'_r, v'_r, $1 \leq r \leq k$ be corresponding column vectors of dimension p_r repartitioned as Q, where, for instance, we denote

$$v'_{r+1} = \begin{bmatrix} v_{q_r+1} \\ \vdots \\ v_{q_{r+1}} \end{bmatrix}, \qquad 0 \leq r < k.$$

Let $y'_1 = g'_1$, $y'_2 = g'_2 - C'_2 y'_1$ where $D'_2 = C'_2 A'_1$. If we let

$$y'_1 = A'_1 w'_1 \tag{2.13}$$

w'_1 may be obtained by an H-process for (2.13). If we denote the H matrices that enter here by $H_1, \cdots, H_{q_1}$ then only H_{q_1} need be inverted and

$$y'_2 = \begin{bmatrix} g_{q_1+1} - D_{q_1+1} w_{q_1} \\ g_{q_1+2} \\ \vdots \\ g_{q_2} \end{bmatrix}.$$

Thus it is seen that only the last vector component w_{q_1} of w'_1 is needed and the reverse sweep given by (2.11) may be omitted. It will also appear later that only w_{q_1} and $H_{q_1}^{-1} H_{q_1-1}$ need be saved for future use. (It should be noted that where an H process is used the D matrices in M'_n are assumed to be non-singular and that this process is feasible only if these D's are easily, if not explicitly, invertible.)

To obtain y'_3 we must compute A'_2, which would generally require the inversion of A'_1. This however is not the case here. Since if we decompose A'_1 by the method of (2.1), (2.2), (2.3) into $A'_1 = L_1 U_1$ then a straightforward computation shows that

$$D'_2 (A'_1)^{-1} E'_1 = D'_2 U_1^{-1} L_1^{-1} E'_1 = \begin{bmatrix} D_{q_1+1} H_{q_1}^{-1} H_{q_1-1} E_{q_1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus A_2' differs from M_2' only in that M_{q_1+1} is replaced by

$$M_{q_1+1} - D_{q_1+1}H_{q_1}^{-1}H_{q_1-1}E_{q_1} .$$

That is A_2' is also of the form (1.1) and has all of its M matrices of the same order p_2. We are therefore reduced to the previous case. We may proceed in this manner until all the y_r', $1 \leq r \leq k$ have been computed and we note that $v_k' = w_k'$. In this last computation, of course, (2.11) will have to be used to get all of v_k'.

To obtain the other v_r' we note from (2.7) that, for $r < k$,

$$v_r' = w_r' - (A_r')^{-1}E_r'v_{r+1}' = w_r' - z_r' ,$$

where we let z_r' be the solution of

$$A_r'z_r' = E_r'v_{r+1}' . \tag{2.14}$$

If the last vector component v_{q_r} of v_r' would be known the other components would be obtainable from the original equations as in (2.11). Since

$$E_r'v_{r+1}' = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_{q_r}v_{q_r+1} \end{bmatrix}$$

if follows from (2.10) and (2.12) that if an H-process were carried out for (2.14) then

$$z_{q_r} = H_{q_r}^{-1}H_{q_r-1}E_{q_r}v_{q_r+1} ,$$

where $H_{q_r}^{-1}H_{q_r-1}$ is the matrix that entered in the computation of w_{q_r}. Thus if these are saved from before, the remaining v_r' are obtained without any further matrix inversions.

Assuming therefore that the work to invert the D-matrices in the M_n' is negligible we see that for a problem involving k sets of M-matrices of equal order only the k matrices H_{q_1}, H_{q_2}, $\cdots$, H_{q_k} need be inverted. Thus in solving Poisson's or the bi-harmonic equation over a region made up of k adjoining rectangles the inversion of at most k relatively small matrices is required. A similar statement applies in higher dimensions.

We note that in the particular case where Q is the discrete Laplacian, where $M_n = J$, $n = 1, 2, \cdots , q$, D_n, E_n are identities and $J = [1, -4, 1]_1^p$, the inverse of H_q can be given explicitly [4]. If we denote the allied Chebyshev polynomials by

$$h_q(a) = \frac{\sinh (q+1)x}{\sinh x}, \qquad 2 \cosh x = a,$$

then $H_q = h_q(J)$. The eigenvalues of J and H_q are given by

$$\lambda_m = 2 \cos \frac{m\pi}{p+1} - 4, \qquad h_q(\lambda_m), \qquad m = 1, 2, \cdots , p,$$

respectively, and the matrix of normalized eigenvectors is

$$G = \left(\frac{2}{p+1}\right)^{1/2} \left\{\sin \frac{km\pi}{p+1}\right\}_{k,m=1}^{p} .$$

If we let

$$L = [h_q(\lambda_1) \,, \cdots \,, h_q(\lambda_p)]$$

be a diagonal matrix then

$$H_q^{-1} = GL^{-1}G.$$

For large problems the H_q may be ill-conditioned. For the above problem the P-condition $P(H_q)$, or ratio of largest to smallest eigenvalue of H_q is

$$P(H_q) \sim \frac{\pi \sinh 6\,(q+1)}{(p+1)\sinh 6 \cdot \sinh\left(\dfrac{q+1}{p+1}\,\pi\right)},$$

so that this may get quite large for large $q = p$. The inversion of H_q may then present severe difficulties. It may however be feasible to break the problem into k groups as indicated above even though all the M_n have the same order.

Since (2.11) represents a marching process there is also the possibility of severe loss in accuracy in the value of v_1 for large q. The value of p does not appear to be an important factor in this loss for the discrete Laplacian problem. In the cases tried for $q = 5$, a 704 code yielded results accurate to at least five significant digits for values of $p = 5, 10, 20, 40$.

3. Criterion for decomposition. A sufficient condition for the validity of the decomposition is similar to that given for the LDU theorem [3]. Let

$$Q_1 = M_1 \,, \qquad Q_2 = \begin{pmatrix} M_1 E_1 \\ D_2 M_2 \end{pmatrix}, \cdots,$$

$$Q_k = [D_n \,, M_n \,, E_n]_1^k \,, \cdots \,, Q_q = Q.$$

If $Q_1 \,, Q_2 \,, \cdots \,, Q_q$ *are non-singular then the decomposition* (2.1)-(2.3) *exists, and is uniquely given by the recursion* (2.4), (2.5). *In this case* $\det Q = \prod_{k=1}^{q} \det A_k$ *and if the* M_k *all have the same order and* $D_2 \,, \cdots \,, D_q$ *are non-singular then* $\det Q = \det H_q$.

The proof follows easily by induction. From the Schur-Frobenius formula

$$\det Q_{q+1} = \det \begin{bmatrix} Q_q & K \\ R & M_{q+1} \end{bmatrix}$$

$$= \det Q_q \cdot \det (M_{q+1} - RQ_q^{-1}K)$$

where $R = (0 \cdots 0 \; D_{q+1})$, $\quad K = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E_q \end{bmatrix}$. From the inductive hypothesis

$$Q_q^{-1} = \begin{bmatrix} A_1^{-1} & & * \\ & \ddots & \\ 0 & & A_q^{-1} \end{bmatrix} \begin{bmatrix} I_1 & & 0 \\ & \ddots & \\ * & & I_q \end{bmatrix},$$

so that $RQ_q^{-1}K = D_{q+1}A_q^{-1}E_q$ and $\det Q_{q+1} = \det Q_q \cdot \det A_{q+1}$. This implies that A_{q+1} is non-singular and also proves the formulas for det Q.

A sufficient condition for $Q_1, \cdots, Q_q$ to be non-singular is, clearly, that for all u

$$\gamma u^T Q u \geq u^T u \tag{3.1}$$

for some non-zero constant γ.

4. Applications. We consider, as an application of the above processes, the problem of solving a certain boundary problem for the Tricomi equation[1]

$$Tu \equiv y u_{xx} - u_{yy} - f(x, y) = 0 \tag{4.1}$$

by a difference approximation given by Friedrichs [5]. These difference approximations for symmetric positive systems have been further investigated by C. K. Chu [1].

The problem for (4.1) is posed for the parallelogram

$$P: | y - x | \leq t, \qquad | x | \leq r; t, \qquad r > 0 \tag{4.2}$$

such that

$$Tu = 0, \qquad (x, y) \in P \tag{4.3}$$

$$u_x + u_y = 0 \quad \text{for} \quad | x - y | = t, \qquad | x | \leq r \tag{4.4}$$

$$u_y = 0 \quad \text{for} \quad x = -r, \qquad | y - x | \leq t. \tag{4.5}$$

No condition is specified on $x = r$.

Since the treatment given by Friedrichs calls for a rectangular region, P is transformed by $\xi = x$, $\eta = y - x$ into the rectangle

$$| \xi | \leq r, \qquad | \eta | \leq t \tag{4.6}$$

and the equation (4.1) is written as the system

$$\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} v_\xi - \begin{bmatrix} y & 1 \\ 1 & 1 \end{bmatrix} v_\eta = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{4.7}$$

where $y = \xi + \eta$, $v = (v_1, v_2)^T$, $v_1 = u_x$, $v_2 = u_y$. It is shown in [1] and [5] that after a premultiplication of (4.7) by a suitable two by two matrix of the form

$$\begin{bmatrix} \rho & y \\ 1 & \rho \end{bmatrix},$$

(4.7) and its boundary conditions can be brought into the required symmetric positive form. That is (4.1), (4.4), (4.5) can be written in the form

$$\tfrac{1}{2}(\alpha^\xi v_\xi + \alpha^\xi v_\eta + (\alpha^\xi v)_\xi + (\alpha^\xi v)_\eta) + \kappa v = g, \tag{4.8}$$

where for a constant ϵ

$$\alpha^\xi = \begin{bmatrix} \rho y & y \\ y & \rho \end{bmatrix}, \qquad \rho = 1 + \epsilon\eta, \tag{4.9}$$

[1]A subscript of x, y, ξ or η indicates partial derivative.

$$\alpha^{\eta} = -\begin{bmatrix} (1+\rho)y & \rho + y \\ \rho + y & 1 + \rho \end{bmatrix}, \tag{4.10}$$

$$g = \begin{bmatrix} \rho f \\ f \end{bmatrix}, \tag{4.11}$$

$$\kappa = -\tfrac{1}{2}(\alpha^{\xi}_{\xi} + \alpha^{\eta}_{\eta}) = \tfrac{1}{2}\begin{bmatrix} 1 + \epsilon y & \epsilon \\ \epsilon & \epsilon \end{bmatrix}. \tag{4.12}$$

As is shown in [1] and [5] the boundary conditions can be written in the form

$$\beta v = \mu v, \tag{4.13}$$

where

$$\beta = -\alpha^{\xi}, \quad \mu = -\alpha^{\xi}_{-} = -\begin{bmatrix} \rho y & y \\ y & \dfrac{2y}{\rho} - \rho \end{bmatrix} \quad \text{for} \quad \xi = -r, \tag{4.14}$$

$$\beta = \alpha^{\xi}, \qquad \mu = \alpha^{\xi}_{+} = \alpha^{\xi} \quad \text{for} \quad \xi = r \tag{4.15}$$

$$\beta = -c^{\eta}, \qquad \mu = -\alpha^{\eta}_{-} \quad \text{for} \quad \eta = -t \quad \text{where} \tag{4.16}$$

$$\alpha^{\eta}_{-} = \frac{1}{\rho - 1}\begin{bmatrix} 2\rho^2 - y(\rho^2 + 1) & (1+\rho)(\rho - y) \\ (1+\rho)(\rho - y) & \rho^2 - 2y + 1 \end{bmatrix}$$

$$\beta = \alpha^{\eta}, \qquad \mu = \alpha^{\eta}_{-} \quad \text{for} \quad \eta = t. \tag{4.17}$$

To obtain the difference equations we divide the intervals $(-r, r)$, $(-t, t)$ by the $2p - 1$, $2q - 1$ points,

$$\xi_i = (i - p)\,\Delta\xi, \qquad 1 \le i \le 2p - 1, \qquad 1 < p, \tag{4.18}$$

$$\eta_i = (j - q)\,\Delta\eta, \qquad 1 \le j \le 2q - 1, \qquad 1 < q, \tag{4.19}$$

respectively, where p and q are odd integers and

$$\Delta\xi = \frac{r}{\rho - 1}, \qquad \Delta\eta = \frac{t}{q - 1}$$

are the respective interval lengths.

According to [5] an equation is written only for the pq odd numbered points (ξ_i, η_i), $i = 2m - 1$, $j = 2n - 1$, $1 \le m \le p$, $1 \le n \le q$. If we write $v_{ij} = v(\xi_i, \eta_i)$, $\alpha^{\xi}_{ij} = \alpha^{\xi}(\xi_i, \eta_i)$ and similarly for the other variables then the difference equations are given for an interior point $1 < m < p$, $1 < n < q$, by

$$\frac{1}{2h}(\alpha^{\xi}_{i+1,j}v_{i+2,j} - \alpha^{\xi}_{i-1,j}v_{i-2,j}) + \frac{1}{2k}(\alpha^{\xi}_{i,j+1}v_{i,j+2} - \alpha^{\xi}_{i,j-1}v_{i,j-2}) + \kappa_{ij}v_{ij} = g_{ij}, \tag{4.20}$$

where $h = 2\,\Delta\xi$, $k = 2\,\Delta\eta$.

For a boundary point at least one of the subscripts $i \pm 1$, $j \pm 1$ falls outside the range prescribed. In that case the α and v of the corresponding term are both evaluated at that boundary point and then replaced according to the rule (4.13). The $2h$ and/or the $2k$ in the difference involved are also replaced by h and/or k, respectively.

As an illustration we consider the equation for the point (ξ_1, η_j), not a corner:

$$\frac{1}{h}(\alpha^{\xi}_{2,j}v_{3,j} - (\alpha^{\xi}_{-})_{ij}v_{1j}) + \frac{1}{2k}(\alpha^{\eta}_{1,j+1}v_{1,j+2} - \alpha^{\eta}_{1,j-1}v_{1,j-2}) + \kappa_{1j}v_{1j} = g_{1j}\,. \tag{4.21}$$

For the corner point (ξ_1, η_1), j, $2k$, $\alpha^{\eta}_{1,j-1}v_{1,j-2}$ are replaced by 1, k and $(\alpha^{\eta}_{-})_{11}v_{11}$, respectively in (4.21). The other types of boundary points are treated in a similar manner.

Since each (m, n) yields one equation we obtain pq equations in the pq unknowns v_{ij}, or $2pq$ scalar equations for $2pq$ scalar unknowns. For each of the equations we note that v_{ij} appears with at most four of its neighbors $v_{i+2,i}$, $v_{i-2,j}$, $v_{i,j-2}$ $v_{i,j+2}$. If we set $v^n_m = v_{ij}$ and denote by v^n the vector with $2p$ scalar components arising from the points (ξ_i, η_{2n-1}) on a horizontal line, the difference equations take the form

$$d^n_m v^{n-1}_m + a^n_m v^n_{m-1} + b^n_m v^n_m + c^n_m v^n_{m+1} + e^n_m v^{n+1}_m = g^n_m \tag{4.22}$$

$$1 \le m \le p, \qquad 1 \le n \le q.$$

This system can be written in the form (1.1), (1.2) where

$$M_n = [a^n_m, b^n_m, c^n_m]^p_{m=1},$$

$$D_n = [0, d^n_m, 0]^p_{m=1},$$

$$E_n = [0, e^n_m, 0]^p_{m=1},$$

$$v = \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^q \end{bmatrix}, \qquad g = \begin{bmatrix} g^1 \\ g^2 \\ \vdots \\ g^q \end{bmatrix},$$

and where for $1 < m < p$, $1 < n < q$

$$d^n_m = -\frac{1}{2k}\alpha^{\eta}_{i,j-1}, \qquad a^n_m = -\frac{1}{2h}\alpha^{\xi}_{i-1,i}$$

$$b^n_m = \kappa_{ij}, \qquad c^n_m = \frac{1}{2h}\alpha^{\xi}_{i+1,i}$$

$$e^n_m = \frac{1}{2k}\alpha^{\eta}_{i,j+1}, \qquad g^n_m = g_{ij}\,.$$

For the boundary points $n = 1, q$, $1 \le m \le p$,

$$d^1_m = 0, \qquad d^q_m = -\frac{1}{k}\alpha^{\eta}_{i,2q-2}$$

$$e^1_m = \frac{1}{k}\alpha^{\eta}_{i,2}, \qquad e^r_m = 0$$

and for $m = 1, p$, $1 \le n \le q$

$$a^n_1 = 0, \qquad a^n_p = -\frac{1}{h}\alpha^{\xi}_{2p-2,i}$$

$$c^n_1 = \frac{1}{h}\alpha^{\xi}_{2,i}, \qquad c^n_p = 0.$$

For $m = 1, p, 1 < n < q$,

$$b_1^n = \kappa_{ij} - \frac{1}{h}(\alpha_-^\xi)_{ij}, \qquad b_p^n = \kappa_{ij} + \frac{1}{h}\alpha_{ij}^\xi$$

and for corner points

$$b_m^n = \kappa_{ij} - \frac{1}{h}(\alpha_-^\xi)_{ij} - \frac{1}{k}(\alpha_-^\eta)_{ij}, \qquad m = 1, \qquad n = 1,$$

$$b_m^n = \kappa_{ij} + \frac{1}{h}\alpha_{ij}^\xi - \frac{1}{k}(\alpha_-^\eta)_{ij}, \qquad m = p, \qquad n = 1,$$

$$b_m^n = \kappa_{ij} - \frac{1}{h}(\alpha_-^\xi)_{ij} + \frac{1}{k}(\alpha_-^\eta)_{ij}, \qquad m = 1, \qquad n = q,$$

$$b_m^n = \kappa_{ij} + \frac{1}{h}\alpha_{ij}^\xi + \frac{1}{k}(\alpha_-^\eta)_{ij}, \qquad m = p, \qquad n = q,$$

where in all cases $i = 2m - 1$, $j = 2n - 1$.

Thus this boundary problem for the Tricomi equation yields a matrix of the form (1.1). It has been shown by Chu [1] that, if ϵ, r, t are properly chosen, e.g., $\epsilon = 1/2$, $r = 1/2$, $t = 1/5$, this matrix satisfies the inequality (3.1). The LU-process is therefore applicable. Since the M_n all have the same order we may in fact use the H-process providing the D_n are non-singular and easily invertible. The D_n are quasi-diagonal, and a simple computation shows that for the above choice of ϵ, r, t, α^η is non-singular.

The above mentioned code was used to solve the problem described above for the choice of

$$f(x, y) = 6yx - 4y^2 + y - 1 - 2x$$

and was run for various values of p and q. The solution to the analytic problem (4.3)-(4.5) can be given explicitly by

$$u(x, y) = (y - x)^2(\tfrac{1}{2} + x) - x/25 \tag{4.23}$$

and for (4.8)-(4.17) it is given by

$$\begin{aligned} v_1 &= \eta^2 - \eta(1 + 2\xi) - 1/25, \\ v_2 &= \eta(1 + 2\xi). \end{aligned} \tag{4.24}$$

For the case of $p = 15$, $q = 15$ the code yielded an answer accurate generally to three significant digits. The values given at $I(0, 0)$, $II(1/2, 0)$ by the code are illustrated by Tables I, II respectively:

$p \times q$		3×3	5×5	7×13	11×7	15×15	
I	v_1	-6.22	-4.07	-4.02	-4.27	-4.04	$\times 10^{-2}$
	v_2	-0.37	0.08	0.01	-0.04	-0.01	
II	v_1	-7.13	-4.96	-4.34	-3.99	-3.99	$\times 10^{-2}$
	v_2	-1.01	0.54	0.14	-0.21	-0.05	

The exact solution given by (4.24) for $\eta = 0$ is $v_1 = -.04$, $v_2 = 0$.

Other problems for the Tricomi equation were run to check the influence of the positivity and boundary conditions. In one case the two by two premultiplier matrix, needed to guarantee "positivity," was omitted. Again an approximate solution was obtained with a slight loss in accuracy. For instance the value at I given by the code was $v_1 = -.0454$, $v_2 = -.0019$, for $p = 11$, $q = 7$.

A problem for the homogeneous Tricomi equation with inhomogeneous boundary conditions $(\mu - \beta)v = f$, for a given f, also yielded results similar to those given above.

Symmetric positive systems for dimensions higher than two may be treated in a similar manner.

As is pointed out in [2] and [4], problems for the bi-harmonic equation can also be put into the form (1.1). The LU-method was carried out for a simply supported rectangular plate and was found to give results accurate to about three significant digits for a 30 by 30 mesh. The running time for this problem was about an hour on the IBM-704.

The above methods may also be combined with iterative or group relaxation methods where each individual group relaxation is done by a direct method. This has already been proposed for a multigroup diffusion problem by Nohel and Timlake [6].

It may also be noted that, in problems where higher order difference schemes are available, the direct methods will require relatively little additional operations and thus one may require fewer mesh points in a given problem.

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—NOTES—

A NOTE ON RIGID BODY DISPLACEMENTS IN THE THEORY OF THIN ELASTIC SHELLS*

By P. M. NAGHDI (*University of California, Berkeley*)

1. Introduction. The stress-strain relations appropriate to the linear theory of thin elastic isotropic shells have long been the subject of numerous investigations. Recently, a *suitable* system of stress-strain relations, which include the effects of both transverse shear deformation and transverse normal stress, has been derived for axisymmetric shells of revolution by E. Reissner [1] and, in general form by the present author [2], where the deformation is referred to the lines of curvature of the middle surface. Also contained in [2] are explicit expressions for the stress resultants $\{N_1, N_2, N_{12}, N_{21}\}$, $\{V_1, V_2\}$, and the stress couples $\{M_1, M_2, M_{12}, M_{21}\}$ in terms of quantities which describe the state of strain, when only the effect of transverse shear deformation is retained (see Eqs. (3.6) of [2]). This latter system of stress-strain relations, in the notation of Ref. [2], is of the form

$$\begin{aligned} N_{12} &= Gh\left[(\gamma_1^0 + \gamma_2^0) + \frac{h^2}{12}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{\gamma_1^0}{R_1} - \delta_1\right)\right], \\ N_{21} &= Gh\left[(\gamma_1^0 + \gamma_2^0) + \frac{h^2}{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left(\frac{\gamma_2^0}{R_2} - \delta_2\right)\right], \\ M_{12} &= \frac{1-\nu}{2} D\left[(\delta_1 + \delta_2) + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\gamma_1^0\right], \\ M_{21} &= \frac{1-\nu}{2} D\left[(\delta_1 + \delta_2) + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\gamma_2^0\right], \text{ etc.,} \end{aligned} \tag{1.1}$$

which satisfy the identity

$$N_{12} + \frac{M_{12}}{R_1} = N_{21} + \frac{M_{21}}{R_2}.$$

In (1.1), R_1 and R_2 are the principal radii of curvature of the middle surface; h is the shell thickness; $D = Eh^3/[12(1 - \nu^2)]$; G is the shear modulus; E and ν are respectively Young's modulus and Poisson's ratio; and the quantities γ_1^0 δ_1, etc., which refer to the components of strain, will be defined presently.

The stress-strain relations derived in [2] have been criticized very recently by Goldenveiser [3], who has asserted that for rigid body displacements alone not all quantities on the right-hand side of (1.1) vanish, which implies that the shell space will not remain stress-free. It is the purpose of the present note to examine the implications of rigid body displacements of the shell space, and this may be regarded as an addition to

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the results in [2]. Although attention is confined to shells whose stress-strain relations are defined by (1.1), the extension to the case in which the effects of both transverse shear deformation and normal stress are included will be apparent and leads to the same conclusion. It is shown that in the absence of surface loads and body forces, contrary to the assertion made in [3], for rigid body displacements all of the stress resultants and the stress couples vanish; and conversely, the vanishing of these resultants corresponds in fact to the most general form of the rigid body displacements of the shell space.

2. Preliminary background. In order to make the following analysis reasonably self-contained, we recall that the position vector of a generic point in the shell space is defined by

$$\mathbf{R}(\xi_1, \xi_2, \zeta) = \mathbf{r}(\xi_1, \xi_2) + \zeta\mathbf{n} \tag{2.1}$$

where $\mathbf{r}$ is the position vector of the corresponding point (with coordinates ξ_1 and ξ_2) on the middle surface, and $-h/2 \leqq \zeta \leqq h/2$ is measured along the unit normal $\mathbf{n}$ to the middle surface. Further, we denote by $\mathbf{t}_1$ and $\mathbf{t}_2$ ($\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{n}$ form a right-handed system), the unit tangent vectors to the ξ_1- and ξ_2-parametric curves which will be referred to the lines of curvature on the middle surface ($\zeta = 0$).

The displacement vector, as assumed in [2] and which leads to the results (1.1), is given by

$$\mathbf{U}(\xi_1, \xi_2, \zeta) = \mathbf{u}(\xi_1, \xi_2) + \zeta\boldsymbol{\beta}(\xi_1, \xi_2), \tag{2.2a}$$

where

$$\mathbf{u} = u_1\mathbf{t}_1 + u_2\mathbf{t}_2 + w\mathbf{n}, \qquad \boldsymbol{\beta} = \beta_1\mathbf{t}_1 + \beta_2\mathbf{t}_2 . \tag{2.2b}$$

With the notation $\partial/\partial\xi_1(\ ) \equiv (\ )_{,1}$ the quantities γ_1^0, δ_1, etc. in (1.1), (or in Eqs. (3.6) of [2]), may be conveniently written as

$$\begin{aligned}
\epsilon_1^0 &= \frac{\mathbf{u}_{,1}\cdot\mathbf{r}_{,1}}{\alpha_1^2}, & \epsilon_2^0 &= \frac{\mathbf{u}_{,2}\cdot\mathbf{r}_{,2}}{\alpha_2^2},\\
\gamma_1^0 &= \frac{\mathbf{u}_{,1}\cdot\mathbf{r}_{,2}}{\alpha_1\alpha_2}, & \gamma_2^0 &= \frac{\mathbf{u}_{,2}\cdot\mathbf{r}_{,1}}{\alpha_2\alpha_1},\\
\alpha_1\gamma_{1\zeta} &= \boldsymbol{\beta}\cdot\mathbf{r}_{,1} + \mathbf{u}_{,1}\cdot\mathbf{n}, & \alpha_2\gamma_{2\zeta}^0 &= \boldsymbol{\beta}\cdot\mathbf{r}_{,2} + \mathbf{u}_{,2}\cdot\mathbf{n},\\
\kappa_1 &= \frac{\boldsymbol{\beta}_{,1}\cdot\mathbf{r}_{,1}}{\alpha_1^2}, & \kappa_2 &= \frac{\boldsymbol{\beta}_{,2}\cdot\mathbf{r}_{,2}}{\alpha_2^2},\\
\delta_1 &= \frac{\boldsymbol{\beta}_{,1}\cdot\mathbf{r}_{,2}}{\alpha_1\alpha_2}, & \delta_2 &= \frac{\boldsymbol{\beta}_{,2}\cdot\mathbf{r}_{,1}}{\alpha_2\alpha_1},
\end{aligned} \tag{2.3}$$

where (α_1^2, α_2^2) are the non-vanishing components of the middle surface metric.

3. Rigid body displacements of the shell space. Let, in the absence of surface loads and body forces, the displacements be specified by

$$\mathbf{U} = \mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{R}, \tag{3.1a}$$

which, by (2.1), may also be written as

$$\mathbf{U} = (\mathbf{u}_0 + \boldsymbol{\omega} \times \mathbf{r}) + \zeta(\boldsymbol{\omega} \times \mathbf{n}), \tag{3.1b}$$

where $\mathbf{u}_0$ and the rotation $\boldsymbol{\omega}$ are constant vectors. The displacement vector defined by (3.1) represents the most general form of rigid body displacements of the shell space,

and includes as special cases purely rigid body translation and purely rigid body rotation.

Returning now to our main objective, we substitute (3.1) into the first six of (2.4) to obtain

$$
\begin{aligned}
\epsilon_1^0 &= \frac{1}{\alpha_1^2}\mathbf{r}_{,1}\cdot(\boldsymbol{\omega}\times\mathbf{r}_{,1}) = 0,\\
\gamma_1^0 &= \frac{1}{\alpha_1\alpha_2}\mathbf{r}_{,2}\cdot(\boldsymbol{\omega}\times\mathbf{r}_{,1}) = \boldsymbol{\omega}\cdot\mathbf{n},\\
\gamma_{1\zeta}^0 &= \frac{1}{\alpha_1}[\mathbf{r}_{,1}\cdot\boldsymbol{\omega}\times\mathbf{n} + \mathbf{n}\cdot\boldsymbol{\omega}\times\mathbf{r}_{,1}],\\
&= \frac{1}{\alpha_1}[\mathbf{r}_{,1}\cdot\boldsymbol{\omega}\times\mathbf{n} - \mathbf{r}_{,1}\cdot\boldsymbol{\omega}\times\mathbf{n}] = 0,
\end{aligned}
\tag{3.2a}
$$

and similarly,

$$
\epsilon_2^0 = 0, \qquad \gamma_2^0 = -\boldsymbol{\omega}\cdot\mathbf{n}, \qquad \gamma_{2\zeta}^0 = 0. \tag{3.2b}
$$

Likewise, substitution of (3.1) into the remaining expressions of (2.4), with the aid of $\mathbf{n}_{,1} = \mathbf{r}_{,1}/R_1$ and $\mathbf{n}_{,2} = \mathbf{r}_{,2}/R_2$, yields

$$
\begin{aligned}
\kappa_1 &= \frac{1}{\alpha_1^2}\boldsymbol{\omega}\times\mathbf{n}_{,1}\cdot\mathbf{r}_{,1} = \frac{1}{\alpha_1^2}\boldsymbol{\omega}\cdot\mathbf{n}_{,1}\times\mathbf{r}_{,1} = 0,\\
\delta_1 &= \frac{1}{\alpha_1\alpha_2}\boldsymbol{\omega}\times\mathbf{n}_{,1}\cdot\mathbf{r}_{,2} = \frac{1}{\alpha_1\alpha_2}\boldsymbol{\omega}\cdot\frac{\mathbf{r}_{,1}.\mathbf{r}_{,2}}{R_1},\\
&= \frac{\boldsymbol{\omega}\cdot\mathbf{n}}{R_1},
\end{aligned}
\tag{3.3a}
$$

and similarly,

$$
\kappa_2 = 0, \qquad \delta_2 = -\frac{\boldsymbol{\omega}\cdot\mathbf{n}}{R_2}. \tag{3.3b}
$$

Next, substituting (3.2) and (3.3) into (1.1) and observing that

$$
\begin{gathered}
\gamma_1^0 + \gamma_2^0 = \frac{\gamma_1^0}{R_1} - \delta_1 = \frac{\gamma_2^0}{R_2} - \delta_2 = 0\\
(\delta_1+\delta_2) + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\gamma_1^0 = (\delta_1+\delta_2) + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\gamma_2^0 = 0,
\end{gathered}
\tag{3.4}
$$

it follows that in the absence of surface loads and body forces, all of the resultants in (1.1) vanish. Furthermore, it may be easily verified that when in the absence of surface loads and body forces all resultants are set equal to zero, the solution of differential equations (1.1) and (2.3), with an appeal to the uniqueness theorem, is indeed given by (3.1).

Note added in proof: Professor Goldenweiser has recently informed the author that he is in agreement with the content of this note.

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A NOTE CONCERNING HYPERBOLIC EQUATIONS WITH CONSTANT COEFFICIENTS*

By JAMES F. HEYDA (*Aircraft Nuclear Propulsion Department, General Electric Co.*)

The canonical form of the linear, homogeneous hyperbolic partial differential equation in two independent variables is

$$u_{xy} + au_x + bu_y + cu = 0, \tag{1}$$

where a, b, c are functions of x, y. The constant coefficients case, frequently met in applitions, can be reduced to the simpler form

$$v_{xy} - \lambda v = 0, \qquad \lambda = ab - c \tag{2}$$

with λ constant, through the substitution

$$v(x, y) = u \exp (bx + ay). \tag{3}$$

The solution of the Cauchy problem for an equivalent form of (2) with v and $\partial v/\partial n$ prescribed along C is worked out explicitly by Copson [1] using the methods of M. Riesz.

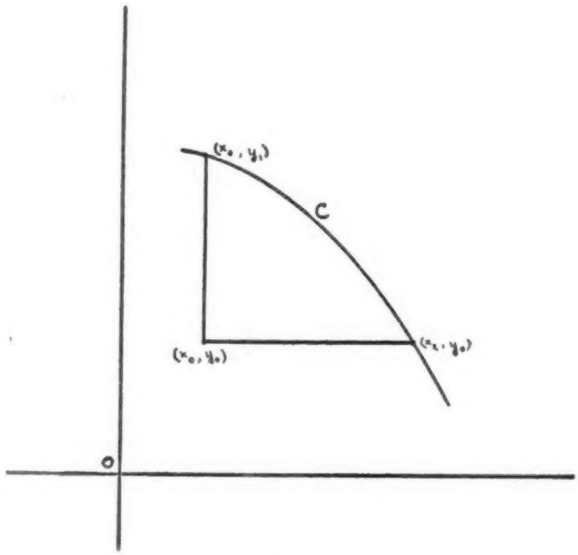

Fig. 1

The form of Copson's solution is the same as that which one would obtain by using Riemann's method for solving the same problem. This enables us to identify the Bessel function $I_0(2[\lambda(x - \alpha)(y - \beta)]^{1/2})$ as the Riemann characteristic function for Eq. (2) By a theorem of Le Roux [2], it then follows that the definite integrals

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$$\int_{x_0}^{x} f(\alpha) I_0(2[\lambda(y - y_0)(x - \alpha)]^{1/2})\, d\alpha,$$
$$\int_{y_0}^{y} g(\beta) I_0(2[\lambda(x - x_0)(y - \beta)]^{1/2})\, d\beta \tag{4}$$

will also be solutions of (2) for arbitrary functions $f(\alpha)$ and $g(\beta)$. This enables us to write down directly a solution of (2) for prescribed values of $v(x, y)$ along the characteristic lines $x = x_0$ and $y = y_0$. For we have from (4) by superposition

$$\begin{aligned} v(x, y) = {} & \int_{x_0}^{x} f(\alpha) I_0(2[\lambda(y - y_0)(x - \alpha)]^{1/2})\, d\alpha \\ & + \int_{y_0}^{y} g(\beta) I_0(2[\lambda(x - x_0)(y - \beta)]^{1/2})\, d\beta + v(x_0, y_0) I_0(2[\lambda(x - x_0)(y - y_0)]^{1/2}), \end{aligned} \tag{5}$$

where

$$f(x) = \frac{\partial v(x, y_0)}{\partial x}, \qquad g(y) = \frac{\partial v(x_0, y)}{\partial y}, \tag{6}$$

the prescribed functions $v(x, y_0)$, $v(x_0, y)$ being assumed suitably well-behaved to carry out the operations indicated.

As an instance of (5) we remark that it furnishes directly a closed form solution of the boundary value problem treated by Mason [3] in connection with heat transfer in cross-flow and explains the mysterious appearance of $I_0(2[abxy]^{\frac{1}{2}})$ in his solution, arrived at by means of Laplace transform methods.

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Corrections to the paper

THERMAL INSTABILITY OF VISCOUS FLUIDS

Quarterly of Applied Mathematics, XVII, 25–42 (1959)

BY CHIA-SHUN YIH (*University of Michigan*)

Equation (27) should read

$$[\sigma - \Pr (D^2 - b^2 - a^2)]\,(D^2 - a^2) f = -R \Pr D\theta$$

The second sentence following Eq. (27) should then be changed to: "Since Eqs. (26) and (27) would be identical to Eqs. (20) and (21) if $\sigma + b^2$ is replaced by σ, any variation of the flow with y will invariably make the flow more stable, if u_2 remains strictly zero, and the boundary conditions on f itself are not relaxed."

With the periodic variation in the y-direction provided, the boundary conditions

on f itself can be relaxed, and the fluid may be more unstable. This instability in fact corresponds to the root zero of the characteristic equation

$$\tan R^{1/4} = \tanh R^{1/4}$$

given in the paper. The author is indebted to Mr. R. A. Wooding of Cavendish Laboratory of Cambridge for pointing out this most unstable mode.

ON THE SOMMERFELD HALF-PLANE PROBLEM*

By BERNARD A. LIPPMANN (*Lawrence Radiation Laboratory, University of California, Livermore, California*)

Abstract. A simple derivation of the Sommerfeld solution to the problem of the diffraction of a plane, scalar wave by a half-plane is given. The discusson is of interest mainly because of the simplicity of the argument; however, it is felt that the ansatz that forms the core of the argument is probably more generally applicable.

I. The method of deriving the Sommerfeld solution to the problem of the diffraction of a plane, scalar wave by a half-plane, presented here, became apparent during a study of the utility of conformal mapping in diffraction problems. The discussion is presented mainly because of the simplicity of the argument employed to deduce Sommerfeld's well-known result, but, beyond this, there are two other features of interest: first, one feels that the ansatz that constitutes the core of the argument is probably applicable more generally; and second, although the explicit conformal mapping used is trivial, the manner in which the mapping enters into the formulation of the boundary conditions in the ansatz may be suggestive in clarifying the relationship between conformal mapping and diffraction theory.

II. We seek a function, u, that is a solution of the wave equation[1]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)u = 0, \tag{1}$$

corresponding to the incident wave[2]

$$\exp\left[-ikr \cos (\theta - \theta')\right], \tag{2}$$

and satisfying certain other conditions to be stated presently. These conditions, the wave equation, and the form of the incident wave, show that u is a function of $\theta - \theta'$ only; we may therefore allow θ' to approach zero and use

$$u_0 = \exp(-ikr \cos \theta) \tag{3}$$

as the incident wave, providing that, in the final result for u, we replace θ by $\theta - \theta'$ everywhere.

It is instructive to consider the problem simultaneously as it appears in Fig. 1, which shows the actual half-plane, coincident with the positive x-axis, and Fig. 2, which

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[1]We shall use the coordinates x, y; r, θ; and $z = y + iy$, $z^* = x - iy$, as convenience dictates.

[2]In contrast to the usual convention, note that this incident wave is traveling towards the x-axis, from above, at an angle θ'.

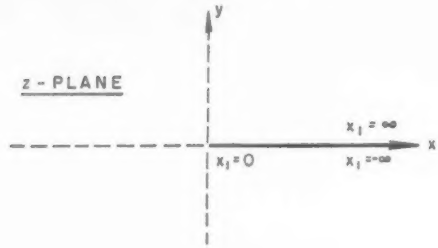

FIG. 1

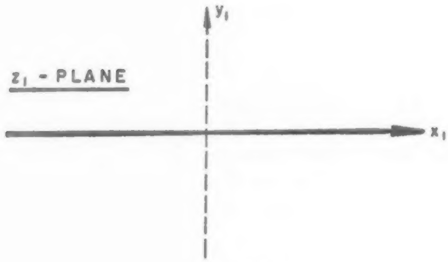

FIG. 2

shows the half-plane mapped on the entire real axis. (Figure 1 is related to Fig. 2 by $z = z_1^2$.)

We shall assume that θ' has approached zero from positive values of θ', so that the incident wave in Fig. 1 is regarded as coming on the half-plane from above, right, at grazing incidence, and moving in the negative x-direction; in Fig. 2, the same wave, suitably transformed, is incident on the entire real axis from above.

Suppose the boundary condition on the half-plane is $u = 0$; that is, u vanishes all along the real axis in the z_1 plane. This condition may be satisfied as follows. Assume we find a solution of the wave equation that has u_0 as its incident wave and that satisfies *any* boundary condition on the x_1-axis. From this solution we subtract the function obtained by reflecting the incident wave in the x_1-axis; the result is a solution of the wave equation that has the correct source above the x_1-axis and reduces to zero on the x_1-axis.

Thus, the only condition we need impose on u, other than that it is a solution of the wave equation, is that the wave u_0 is incident on the half-plane from above; the boundary condition on the half-plane may be chosen at our convenience.

One simple choice regards the half-plane as "black". That is, the half-plane absorbs the incident wave completely. No scattered wave is required, except at the edge of the half-plane; here, a scattered wave is needed in order to maintain the continuity of the wave function u in going around the half-plane.

This boundary condition means that we must find a (continuous) function u such that:

$$u \text{ satisfies the wave equation.} \tag{A-1}$$

$$u \to u_0 \text{ as } x_1 \to \infty \text{ for } y_1 > 0. \tag{A-2}$$

$$u \to 0 \text{ as } x_1 \to -\infty \text{ for } y_1 > 0. \tag{A-3}$$

We now proceed to satisfy these conditions. Our method is based on the fundamental ansatz:

$$u = u_0 I(x_1), \tag{4}$$

where we require that

$$I(\infty) = 1, \tag{5}$$

and

$$I(-\infty) = 0, \tag{6}$$

a pair of conditions that can be met by putting

$$I(x_1) = \int_{-\infty}^{ax_1} f(\tau)\, d\tau, \tag{7}$$

where a is an arbitrary constant.

(A-2) and (A-3) are now satisfied. Moreover, $f(\tau)$ is still arbitrary, leaving the possibility that it can be chosen to satisfy (A-1).

Since

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{|z_1|^2} \frac{\partial^2}{\partial z_1\, \partial z_1^*},$$

(A-1) demands that

$$\left\{\frac{1}{|z_1|^2} \frac{\partial^2}{\partial z_1\, \partial z_1^*} + k^2\right\} \exp\left[-\frac{ik}{2}(z_1^2 + z_1^{*2})\right] I\left(\frac{a}{2}(z_1 + z_1^*)\right) = 0. \tag{8}$$

From this, we deduce that $\left[f'(\tau) \equiv \dfrac{d}{d\tau} f(\tau)\right]$

$$\frac{f'(ax_1)}{f(ax_1)} = \frac{4ik}{a^2} \cdot ax_1 ;$$

or,

$$f(\tau) = \alpha \exp(i2k\tau^2/a^2). \tag{9}$$

Putting this in (7), adjusting α to satisfy the normalization requirement (5), and using $x_1 = \rho^{1/2} \cos \theta/2$, we obtain:

$$I(x_1) = \int_{-\infty}^{a\rho^{1/2}\cos\theta/2} \left(\frac{2k}{\pi a^2 i}\right)^{1/2} \exp(2ik\tau^2/a^2) = (\pi i)^{-1/2} \int_{-\infty}^{(2k\rho)^{1/2}\cos\theta/2} \exp(i\tau^2)\, dt. \tag{10}$$

The complete solution is now given by making the replacement $\theta \to \theta - \theta'$ in (3), (4), and (10):

$$u(\theta; \theta') = (\pi i)^{-1/2} \exp[-ikr\cos(\theta - \theta')] \int_{-\infty}^{(2k\rho)^{1/2}\cos(\theta-\theta')/2} \exp(i\tau^2)\, d\tau. \tag{11}$$

The solution corresponds to a "black" half-plane, but, following the prescription given earlier, it can easily be transcribed into the solution for a perfectly conducting half-plane ($u = 0$ on the half-plane): In the z_1-plane, reflecting the source in the real axis changes θ_1' to $-\theta_1'$; this transformation replaces θ' by $-\theta'$ in the z-plane. Therefore, the required function is

$$u(\theta; \theta') - u(\theta; -\theta'), \tag{12}$$

since this takes on the value zero for $\theta = 0; 2\pi$.

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